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Two-Loop String Theory on Null Compactifications

Henry C.D. Cove and **Richard J. Szabo**

Department of Mathematics
and
Maxwell Institute for Mathematical Sciences
Heriot-Watt University
Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, U.K.
henry@ma.hw.ac.uk , R.J.Szabo@ma.hw.ac.uk

Abstract

We compute the two-loop contributions to the free energy in the null compactification of perturbative string theory at finite temperature. The cases of bosonic, Type II and heterotic strings are all treated. The calculation exploits an explicit reductive parametrization of the moduli space of infinite-momentum frame string worldsheets in terms of branched cover instantons. Various arithmetic and physical properties of the instanton sums are described. Applications to symmetric product orbifold conformal field theories and to the matrix string theory conjecture are also briefly discussed.

1 Introduction and Summary

Compactifications of string theory along a dimension which is light-like, rather than space-like, are of interest for a variety of reasons (See [54] for reviews of some of the issues addressed in the following). A light-like circle can be gotten from a space-like one by an infinite Lorentz boost [53]. By T-duality, for any light-like compactification radius R the energy spectrum of the rest frame states coincides with that of the uncompactified string theory. These compactifications thereby probe the T-dual string theory in a regime wherein long fundamental string states wrap an almost infinite compact direction. In discrete light-cone quantization (DLCQ) on flat ten-dimensional spacetime, the momentum p^+ along the compact null direction x^- is quantized as

$$p^+ = \frac{N}{R} \tag{1.1}$$

with $N \in \mathbb{N}$, while the light-cone energy p^- is determined by the mass-shell relation

$$p^- = \frac{1}{p^+}\left(L_0 + \overline{L}_0\right) = \frac{1}{p^+}\,H \ . \tag{1.2}$$

Analysis of the Hilbert space shows [23, 20, 28] that free second-quantized Type IIA superstring theory is naturally equivalent to a free superconformal sigma-model on the symmetric product orbifold

$$\mathrm{Sym}^N\left(\mathbb{R}^8\right) = \mathbb{R}^{8N}/S_N \ , \tag{1.3}$$

in that the corresponding conformal field theory vacuum amplitudes coincide in the free string infrared limit $g_s \to 0$.

This equivalence may be given a geometric interpretation by introducing a finite temperature [31, 33]. This is done by further compactifying Euclidean time so that two target space directions are compactified on a torus $\mathbb{T}^2_\tau$ of a particular modulus τ. For the present discussion, the thermodynamic partition function is simply regarded as a generating function for the energy spectrum of free string theory and thermal instabilities such as the gravitational Jeans instability or the stringy Hagedorn transition will be ignored. On the Type IIA side, the one-loop free energy is given by a sum over unramified coverings of the torus $\mathbb{T}^2_\tau$ of degree N [31, 33]. On the superconformal field theory side, the partition function on $\mathbb{T}^2_\tau$ is given by a sum over twisted sectors imposing S_N-twisted boundary conditions on the string embedding fields. An extra summation over elements of S_N is required to define a projection onto the S_N-invariant subspace of the Hilbert space, resulting in a sum over commuting pairs of permutations assuring that the twists in time and space directions commute. The twisted sectors have a natural interpretation in terms of "long" strings formed from "short" fundamental string bits. The partition function from N fundamental single strings are combined together to give the partition function of one long string with a modified modular parameter, i.e. the worldsheet of the long strings is an N-fold cover of the torus. The pertinent combinatorics is summarized by the action of the Hecke operator [23] which maps a modular form into another one with the same weight. The action of the Hecke algebra admits an interpretation in terms of the creation of a long string background along with the addition of short string excitations to it [36]. In this comparison it is of course more natural to work with the grand canonical partition function by taking an ensemble of sigma-models on $\mathrm{Sym}^N(\mathbb{R}^8)$ for all $N \in \mathbb{N}$.

In this paper we will examine this correspondence for the interacting string theory with $g_s > 0$ which arises by relaxing the free string infrared limit. This is obtained by perturbing the

orbifold conformal field theory on (1.3). To leading order, this perturbation is described by the Dijkgraaf-Verlinde-Verlinde (DVV) twist field [22] which perturbs the free Hamiltonian H via the density

$$V_{\rm int} = g_s \sum_{1\leq a<b\leq N} \left(\tau^i\, \Sigma_i \otimes \overline{\tau}^j\, \overline{\Sigma}_j\right)_{a,b} + O\left(g_s^2\right) \ , \tag{1.4}$$

where τ^i, $i = 1,\dots,8$ are the excited bosonic twist fields and Σ_i are the fermionic spin fields. This defines a conformal field of weight $(\frac{3}{2},\frac{3}{2})$ which is the unique least irrelevant perturbation that preserves $Spin(8)$ spacetime rotations and spacetime supersymmetry, and which creates a square-root branch cut in the sigma-model with coordinates $x_a^i - x_b^i$. It intertwines between different topological sectors of the worldsheet theory on $\mathbb{T}^2_\tau$ that are related by a basic splitting and joining interaction between pairs of strings. Thus if we use the Hamiltonian density (1.4) for computing scattering amplitudes via standard perturbation theory, then we should reproduce the conventional perturbative expansion of Type IIA superstring theory [4]. This expectation is supported by the fact [21] that the DVV twist field exactly reproduces the Lorentz-invariant Mandelstam cubic interaction vertex that describes the joining and splitting of Type II strings in light-cone gauge. Analysis of higher-order contact terms reveals that the structure of superstring field theory simplifies when expressed in terms of twist field correlators [21, 48].

In DLCQ string theory at finite temperature, the g-loop free energy receives contributions from only those genus g string worldsheets which are branched covers of the spacetime torus $\mathbb{T}^2_\tau$ [33]. This gives a partial discretization of the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces which reduces its complex dimension from $3g-3$ to $2g-3$ (from 1 to 0 for $g=1$). Thus perturbative string theory can be formulated entirely in terms of covering Riemann surfaces, a scenario familiar from the Gross-Taylor string expansion of the two-dimensional Yang-Mills theory [35, 17]. In this paper we will work out explicitly the two-loop free energy which is computed from genus two worldsheets which are branched covers of $\mathbb{T}^2_\tau$. A surface of genus two can be realized as a double cover of the complex plane with three distinct branch cuts. Since any elliptic curve is a double cover of the plane with two branch cuts, a genus two surface can be built from two tori by identifying one of their branch cuts and gluing them together along the cut. This means that the two-loop partition function should coincide with the correlator of two twist fields (1.4) in the symmetric orbifold conformal field theory on $\mathbb{T}^2_\tau$. Such a coincidence is not entirely surprising, given that correlation functions of twist fields can be computed by means of free string partition functions on the appropriate covering space [44]. Indeed, many aspects of string theory (at zero temperature) can be recovered from the sigma-model with target space (1.3) [4, 40]. However, while the twist field correlator appears to be expressed in terms of branch point loci, the DLCQ string free energy is naturally parametrized in terms of pinching parameters corresponding to the sewing construction of the genus two cover from an unramified covering of the spacetime torus and an auxilliary torus. This suggests an interpretation of the correlation function $\langle V_{\rm int}\, V_{\rm int}\rangle$ as the overlap between a long string state and a fundamental string state, a result which is consistent with the physical interpretation of the Hecke algebra mentioned above. We will leave the detailed comparison of our results to twist field correlators for future work. Here we perform the explicit calculations required in DLCQ string theory to check these and other correspondences, as well as to elucidate general higher-loop aspects of perturbative string theory in the branched cover instanton representation.

Our results also pertain to some other contexts. Foremost among these is a two-loop order check of the matrix string theory conjecture [22, 50, 8]. With the notations set above, ma-

trix strings at finite temperature are described by maximally supersymmetric Yang-Mills gauge theory on $\mathbb{T}^2_\tau$ with gauge group $U(N)$ [31]. The string coupling constant g_s is related to the Yang-Mills coupling constant $g_{\rm YM}$ through

$$g_s = \frac{1}{g_{\rm YM}\,\ell_s} \ , \tag{1.5}$$

where $\ell_s = \sqrt{\alpha'}$ is the string length. Thus at weak string coupling the supersymmetric Yang-Mills theory becomes strongly coupled and approaches a superconformal infrared fixed point which is believed to be the supersymmetric orbifold sigma-model discussed above [22]. The DVV twist field (1.4) is then the least irrelevant operator which preserves the Yang-Mills supersymmetry. The string degrees of freedom which emerge in the perturbative string limit $g_s \to 0$ are simultaneous eigenvalues of the matrices. At finite temperature the matrices are defined on the torus $\mathbb{T}^2_\tau$ and their eigenvalues, which solve polynomial equations, are functions on branched covers of $\mathbb{T}^2_\tau$ [31]. At leading order, the thermodynamic free energy of matrix string theory arising from summing over unbranched covers coincides with the DLCQ free energy [31, 33]. At next order, instantons interpolate between initial and final states of strings through the genus two cover of the torus [58, 13, 14, 16, 12, 29]. By finding the BPS instantons which represent the branch points [14, 42] and the correct instanton measure, the matrix model partition function should coincide with the twist field correlator of the symmetric orbifold model in the strong coupling $g_{\rm YM} \to \infty$ limit which is computed by our genus two DLCQ free energy. For this comparison, the most appropriate DLCQ theory is that of the Green-Schwarz superstring at finite temperature with worldsheets of long strings.

Our detailed computations and results could also shed further light on aspects of more complicated symmetric orbifold conformal field theories. An important example is when the orbifold target space is taken to be $\mathrm{Sym}^N(\mathcal{M})$ with $\mathcal{M} = \mathrm{K3}$ or $\mathcal{M} = \mathbb{T}^4$ [1]. With $N = k\,n$, a particular deformation of the superconformal field theory is the sigma-model on the moduli space of k instantons in $U(n)$ gauge theory on $\mathcal{M}$ which is believed to be dual, via the AdS/CFT correspondence, to Type II string theory on the background geometry $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathcal{M}$. The primary evidence for these particular correspondences comes from the matching of their BPS spectra. Finally, from a mathematical perspective our results are related to the computation of elliptic genera [23, 20] and topological Euler characteristics of Hilbert schemes [30], which in the case $\mathcal{M} = \mathrm{K3}$ is related to generalized Kac-Moody algebras [41, 34].

The organisation of the remainder of this paper is as follows. In Section 2 we review the basic arguments establishing that DLCQ string theory at finite temperature is a theory of branched coverings of a torus [33]. We also outline some generic aspects of a certain reduction technique for the Hurwitz moduli space of branched covers which will be central to our analysis throughout this paper. We conclude by reviewing the one-loop calculation [31, 33] in this light for later comparison with the two-loop results.

In Section 3 we begin the construction of the two-loop free energy. We present an explicit description of the moduli space of genus two branched covers using a particular reduction technique. As an example, we compute the bosonic free energy in terms of genus one theta-functions of the elliptic curve $\mathbb{T}^2_\tau$. While bosonic string theory cannot emerge from a gauge theory (since the necessary supersymmetric cancellations of fluctuation determinants do not occur), this calculation can be compared to the bosonic sigma-model with target space $\mathrm{Sym}^N(\mathbb{R}^{24})$ and the

interaction density (1.4) modified by replacing $\tau^i\,\Sigma_i$ with the unexcited twist field [52]

$$\sigma = \prod_{i=1}^{24} \sigma^i \tag{1.6}$$

of dimension $\frac{3}{2}$ having the supersymmetry variation $G^{\dot a}_{-1/2}(\sigma\,\Sigma_{\dot a}) = \tau^i\,\Sigma_i$.

In Section 4 we compute the two-loop superstring free energy. Our calculation draws heavily on recent progress [19] in two-loop superstring perturbation theory in the NSR formalism which yields explicit unambiguous expressions for the chiral superstring measure in terms of genus two modular forms. With the appropriate modification of the genus two GSO projection at finite temperature [5], we find a formula for the superstring free energy in terms of theta-functions on $\mathbb{T}^2_\tau$. In Section 5 we perform the analogous calculation for the heterotic string. In this case the pertinent conformal field theory is the supersymmetric heterotic sigma-model defined on the symmetric product orbifold [52, 43]

$$\mathrm{Sym}^N\big(\mathbb{R}^8\times G\big) = \big(\mathbb{R}^8\times G\big)^N/S_N \ltimes (\mathbb{Z}_2)^N \tag{1.7}$$

for the heterotic gauge group G. The interaction density (1.4) should be modified to contain the bosonic twist field $\overline{\sigma}$ given by (1.6) in the right-moving sector and the supersymmetric twist field $\tau^i\,\Sigma_i$ in the left-moving sector. The relevant gauge dynamics is conjectured to be governed by heterotic matrix string theory [52, 37, 7, 43, 15], i.e. two-dimensional supersymmetric Yang-Mills theory with chiral anomaly-free matter fields and gauge group $O(N)$.

Our formulas for the free energies, while in principle being explicit, are quite complicated. In Section 6 we consider various degeneration limits of the genus two covers in which these expressions drastically simplify, and hence elucidate various arithmetic and physical properties of our amplitudes. We find the appropriate modification of the action of the Hecke algebra for twist field correlators. In a certain collapsing limit, we also find effective one-loop string theories which resemble non-supersymmetric strings on particular $\mathbb{Z}_2$-orbifolds. In another collapsing limit, the partition function resembles the one-loop instanton sum over long string configurations. Finally, in Appendix A we present an alternative reductive description of the moduli space of genus two branched covers which may be of independent interest and use in other applications, while Appendix B contains some technical details of the calculations performed in the main text.

2 String Worldsheets in Light-Like Compactifications

In this section we shall describe the general set-up for the calculation that we will undertake. The main new technical tool we introduce is the method of reduction which works for any branched covering of a Riemann surface by another Riemann surface. We shall also review the well-known one-loop calculation in this new light for the purpose of later comparison and because it will play a role in some of our analysis at two-loops in subsequent sections.

2.1 Discrete Light-Cone Quantization of String Theory at Finite Temperature

Consider the discrete light-cone quantization (DLCQ) of Type II superstring theory at finite temperature using the Polyakov path integral [33]. We work throughout in the Neveu-Schwarz-

Ramond formalism. In string perturbation theory, the gauge-fixed action in the conformal gauge and in Euclidean spacetime at genus g is $S[X] + \overline{S[X]} + S[B,C] + \overline{S[B,C]}$, where

$$S[X] + S[B,C] = \frac{1}{4\pi\,\alpha'} \int\limits_{\Sigma_g} \mathrm{d}^2 z\ \left(\tfrac{1}{2}\,|\partial x^\mu|^2 + \psi_\mu\,\overline{\partial}\psi^\mu + b\,\overline{\partial}c + \beta\,\overline{\partial}\gamma\right) \tag{2.1}$$

and $\sqrt{\alpha'}$ is the string scale. Here $X = (x^\mu, \psi^\mu)_{\mu=0}^9$ denotes the spacetime matter fields, while B and C denote the b,β and c,γ ghost fields, respectively, with (b,c) the spin $(2,1)$ conformal ghost fields and (β,γ) the spin $(\frac{3}{2},\frac{1}{2})$ superconformal ghost fields. The worldsheet is an oriented compact Riemann surface Σ_g of genus g whose first homology group is generated by a set of canonical one-cycles $\boldsymbol{a} = (a_i)_{i=1}^g$, $\boldsymbol{b} = (b_i)_{i=1}^g$ with intersection numbers

$$a_i \cap a_j = b_i \cap b_j = 0 \qquad , \qquad a_i \cap b_j = -b_j \cap a_i = \delta_{ij}\ . \tag{2.2}$$

This intersection form is summarized by the matrix

$$\mathsf{J}_g = \begin{pmatrix} \mathbf{0}_g & \mathbb{1}_g \\ -\mathbb{1}_g & \mathbf{0}_g \end{pmatrix} \tag{2.3}$$

with $\mathsf{J}_g^2 = -\mathbb{1}_g$ which makes $H_1(\Sigma_g,\mathbb{R})$ into a symplectic vector space. The first cohomology group $H^{1,0}(\Sigma_g,\mathbb{C})$ is spanned by a set of holomorphic one-differentials $\boldsymbol{\omega} = (\omega_i)_{i=1}^g$ which have the period normalizations

$$\oint\limits_{a_i} \omega_j = \delta_{ij} \qquad , \qquad \oint\limits_{b_i} \omega_j = \Omega_{ij}\ , \tag{2.4}$$

where Ω is the period matrix of Σ_g which lives in the Siegal upper half-plane $\mathcal{H}_g$ of $g \times g$ complex-valued, symmetric matrices of positive definite imaginary part. We shall throughout write $\Omega = \Omega_1 + \mathrm{i}\,\Omega_2$, where Ω_1 and Ω_2 are real-valued symmetric matrices with $\Omega_2 > 0$.

The DLCQ and finite temperature conditions are imposed by two spacetime compactifications which may be described by the respective identifications

$$\begin{aligned} \left(x^0\,,\,\boldsymbol{x}\,,\,x^9\right) &\sim \left(x^0 + \sqrt{2}\,\pi\,\mathrm{i}\,R\,,\,\boldsymbol{x}\,,\,x^9 - \sqrt{2}\,\pi\,R\right)\,, \\ \left(x^0\,,\,\boldsymbol{x}\,,\,x^9\right) &\sim \left(x^0 + \beta\,,\,\boldsymbol{x}\,,\,x^9\right) \end{aligned} \tag{2.5}$$

where R is the radius of the light-cone in Minkowski space, and $\beta = 1/k_{\mathrm{B}}\,T$ with T the temperature and k_{B} the Boltzmann constant. The corresponding path integral, with the appropriate modification of the GSO projection to make spacetime fermions anti-periodic under $x^0 \to x^0 + \beta$, then computes the thermodynamic free energy of the superstring. The compactification conditions induce quantized zero modes in the mode expansions of the bosonic string embedding fields x^μ corresponding to the wrappings of the various homology cycles of Σ_g around the compact spacetime dimensions. The windings of $(\boldsymbol{a},\boldsymbol{b})$ around the light-cone are labelled by integers $(\boldsymbol{p},\boldsymbol{q})$ and by $(\boldsymbol{n},\boldsymbol{m})$ around the time direction. Apart from the modification of the GSO projection by the temperature winding numbers $(\boldsymbol{n},\boldsymbol{m})$, the only place that these integers appear are as zero mode soliton contributions to the bosonic matter part of the action (2.1). In the path integral one should sum over all possible topological winding sectors. The crucial point is that the action (2.1) depends linearly in a purely imaginary form on the set of integers $(\boldsymbol{p},\boldsymbol{q})$, which when summed thereby produce periodic Dirac delta-functions.

In this way, the finite-temperature, DLCQ superstring free energy (per unit spacetime volume) at genus g is found to be given by [33]

$$
\begin{aligned}
F_g \;&=\; -g_s^{2g-2}\,\nu^{2g}\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^g}\;\sum_{\boldsymbol{r},\boldsymbol{s}\in\mathbb{Z}^g}\;\int\limits_{\mathcal{F}_g}\mathrm{d}\mu_g\begin{bmatrix}\boldsymbol{n}\\ \boldsymbol{m}\end{bmatrix}\big(\Omega\,,\,\overline{\Omega}\,\big)\;\det\Omega_2\;\,\mathrm{e}^{\,-\frac{\beta^2}{4\pi\,\alpha'}\,(\Omega\boldsymbol{n}-\boldsymbol{m})^\dagger\,(\Omega_2)^{-1}(\Omega\boldsymbol{n}-\boldsymbol{m})}\\
&\qquad\times\prod_{j=1}^g\delta\Big(\sum_{i=1}^g(n_i+\mathrm{i}\,\nu\,r_i)\,\Omega_{ij}-(m_j+\mathrm{i}\,\nu\,s_j)\Big)\;, \qquad (2.6)
\end{aligned}
$$

where g_s is the string coupling constant and

$$
\nu=\frac{4\pi\,\alpha'}{\sqrt{2}\,\beta\,R}\;. \qquad (2.7)
$$

The sums in (2.6) go over all four g-vectors of integers $\boldsymbol{m},\boldsymbol{n},\boldsymbol{r},\boldsymbol{s}$ such that the period matrix Ω is in a fundamental modular domain $\mathcal{F}_g$. The modular invariant, genus g superstring measure on moduli space $\mathcal{M}_g$ is denoted $\mathrm{d}\mu_g[{\boldsymbol{n}\atop\boldsymbol{m}}](\Omega,\overline{\Omega}\,)$, and its dependence on the temperature winding integers arises from the modification of the sum over worldsheet spin structures that breaks supersymmetry in the finite temperature theory [5]. The expression (2.6) contains a constraint on the Riemann surfaces Σ_g which contribute to the partition function. As we now explain, it is equivalent to summing over all genus g branched covers Σ_g of the torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$ whose Teichmüller parameter is $\mathrm{i}\,\nu$ [33].

Let $f:\Sigma_g\to\mathbb{T}^2_{\mathrm{i}\,\nu}$ be a holomorphic map, i.e. a branched covering. The covering map induces a homomorphism between the first homology groups via the push-forward

$$
f_*\;:\;H_1(\Sigma_g,\mathbb{Z})\;\longrightarrow\;H_1(\mathbb{T}^2_{\mathrm{i}\,\nu},\mathbb{Z})\;. \qquad (2.8)
$$

Choosing canonical homology bases $(\boldsymbol{a},\boldsymbol{b})$ and (α,β) of the covering space Σ_g and the base space $\mathbb{T}^2_{\mathrm{i}\,\nu}$, respectively, this homomorphism can be written explicitly in terms of an integral $2\times 2g$ matrix

$$
\mathsf{M}=\begin{pmatrix}\boldsymbol{n}&\boldsymbol{m}\\ \boldsymbol{r}&\boldsymbol{s}\end{pmatrix} \qquad (2.9)
$$

of maximal rank acting on the homology generators of the base torus as

$$
f_*\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}=\mathsf{M}^\top\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\;. \qquad (2.10)
$$

Similarly, the covering map induces through pull-back a homomorphism $f^*:H^{1,0}(\mathbb{T}^2_{\mathrm{i}\,\nu},\mathbb{C})\to H^{1,0}(\Sigma_g,\mathbb{C})$ on the first cohomology groups, and there exists a complex $g\times 1$ matrix H of maximal rank which relates the normalized holomorphic differentials $\boldsymbol{\omega}$ and ω on Σ_g and $\mathbb{T}^2_{\mathrm{i}\,\nu}$ by

$$
\omega=\mathsf{H}^\top\boldsymbol{\omega}\;. \qquad (2.11)
$$

The matrix H can be used to give an explicit formula for the covering map as $f(z)=(\Psi\circ\mathfrak{A})(z):=\mathsf{H}^\top\mathfrak{A}(z)\mod\mathbb{Z}\oplus\mathrm{i}\,\nu\,\mathbb{Z}$, where $\mathfrak{A}$ is the Abel map embedding Σ_g into its Jacobian variety $\mathrm{Jac}(\Sigma_g):=\mathbb{C}^g/\mathbb{Z}^g\oplus\Omega\,\mathbb{Z}^g$. This characterization exploits the fact that the Jacobian variety of the curve Σ_g represents a fibration over the elliptic curve $\mathbb{T}^2_{\mathrm{i}\,\nu}$ with the commutative diagram

$$
\begin{tikzcd}
\Sigma_g \arrow[r, "\mathfrak{A}"] \arrow[dr, "f"'] & \mathrm{Jac}(\Sigma_g) \arrow[d, "\Psi"] \\
 & \mathbb{T}^2_{\mathrm{i}\,\nu}
\end{tikzcd}\;. \qquad (2.12)
$$

Furthermore, by computing the α and β periods of both sides of (2.11) we arrive at the matrix equality

$$\mathsf{H}^\top\,(\mathbb{1}_g,\Omega) = (1,\,\mathrm{i}\,\nu)\,\mathsf{M}\ . \tag{2.13}$$

By using the explicit form (2.9) one finds $\mathsf{H} = \boldsymbol{n} + \mathrm{i}\,\nu\,\boldsymbol{r}$ and the equation (2.13) is equivalent to the period matrix constraint in (2.6).

The degree $\deg(f)$ of the covering map can be computed from the Hopf condition [45]

$$\mathsf{M}\,\mathsf{J}_g\,\mathsf{M}^\top = \deg(f)\ \mathsf{J}_1 \tag{2.14}$$

giving $\deg(f) = \boldsymbol{n}\cdot\boldsymbol{s} - \boldsymbol{m}\cdot\boldsymbol{r}$. The computation of the periods in (2.11) leads to a homogeneous linear equation in the variables $\boldsymbol{n},\boldsymbol{m},\boldsymbol{r},\boldsymbol{s}$ and $\mathrm{i}\,\nu$ which has the compatibility condition

$$\det\begin{pmatrix} \mathbb{1}_g & & \Omega \\ \dots & \dots & \dots \\ & \mathsf{M} & \end{pmatrix} = 0\ . \tag{2.15}$$

The formula (2.15) restricts the allowed Riemann period matrices of the curve Σ_g to lie in a Humbert variety inside $\mathcal{H}_g$.

2.2 Weierstrass-Poincaré Reduction

The superstring integration measure $\mathrm{d}\mu_g[{}^{\boldsymbol{n}}_{\boldsymbol{m}}](\Omega,\overline{\Omega}\,)$ is invariant under the mapping class group of the covering Riemann surface. This invariance can be exploited in a manner which simplifies explicit calculations. The Siegal modular group of Σ_g is $Sp(2g,\mathbb{Z})$ which preserves the intersection form (2.3). With respect to the canonical basis of $\mathbb{R}^{g,g}$, it consists of matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\quad,\quad D^\top\,B - B^\top\,D = C^\top\,A - A^\top\,C = 0\quad,\quad A^\top\,D - C^\top\,B = \mathbb{1}_g\ . \tag{2.16}$$

This group acts on a canonical homology basis of Σ_g as

$$\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \longmapsto \begin{pmatrix} \boldsymbol{a}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix}\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}\ . \tag{2.17}$$

The temperature winding integers $(\boldsymbol{n},\boldsymbol{m})$ transform in the same way as $(\boldsymbol{a},\boldsymbol{b})$, and so do the integers $(\boldsymbol{r},\boldsymbol{s})$ which come from compactification of the light-cone. Using (2.16) the inverse of the transformation (2.17) is easily found to be

$$\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} A^\top & -C^\top \\ -B^\top & D^\top \end{pmatrix}\begin{pmatrix} \boldsymbol{a}' \\ \boldsymbol{b}' \end{pmatrix}\ . \tag{2.18}$$

The projective modular group $PSp(2g,\mathbb{Z})$ acts naturally on the Siegal upper half-plane $\mathcal{H}_g$ of $g\times g$ period matrices as

$$\Omega \ \longmapsto\ \Omega' = (A\,\Omega + B)\,(C\,\Omega + D)^{-1}\ . \tag{2.19}$$

For genera $g = 1,2,3$, the moduli space of Σ_g is "almost" given by [49]

$$\mathcal{M}_g = \mathcal{H}_g/PSp(2g,\mathbb{Z})\ . \tag{2.20}$$

For $g \geq 4$ the period matrix can still be used to parametrize moduli space by imposing Schottky relations on Ω. Note that the delta-function constraint of (2.6) is modular covariant.

We can now use the technique of reduction to simplify the constraint equation on Ω in (2.6) before solving it. The technique of reduction is described in [45] (see also [39, 11]) for the general case of coverings of Riemann surfaces of arbitrary genus. Reduction comprises the use of modular transformations on the base and cover in order to make a change in homology basis so that the number of homology cycles on the cover which effectively wind around the base is reduced. It yields a convenient canonical form for the underlying algebraic curve Σ_g which can be thought of as a higher genus version of the canonical Weierstrass parametrization of an elliptic curve. In the present case the periods meet the conditions of the fundamental Weierstrass-Poincaré theory of the complete reducibility of abelian integrals to lower genera [45], which deals with general abelian tori and their associated theta-functions. The main idea is that the curve Σ_g, being a covering of a torus, has a rich group of automorphisms which leads to a decomposition of its Jacobian variety. By considering the curve as a spectral variety, one can thereby reduce the corresponding theta-functions to lower genera. Furthermore, the technique greatly simplifies the analysis of moduli space integrals such as (2.6) by extending the usual Rankin-Selberg method of "unwrapping" modular integrals [46].

We will use the Poincaré reducibility theorem applied to the special case of a covering $f : \Sigma_g \to \mathbb{T}^2_{\,\mathrm{i}\,\nu}$. It relies [45] on the existence of a Frobenius normal form for the $2 \times 2g$ integral matrix (2.9), satisfying the Hopf condition (2.14), given by

$$\mathsf{M} = \mathsf{S}\,\mathsf{P}\,\mathsf{T} \tag{2.21}$$

where S and T are, respectively, 2×2 and $2g\times 2g$ symplectic unimodular matrices. The Poincaré normal form is given by the $2 \times 2g$ matrix

$$\mathsf{P} = r \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & & 0 & 0 & 0 & \dots & 0 \\ 0 & s & 0 & \dots & 0 & & t & 0 & 0 & \dots & 0 \end{pmatrix} , \tag{2.22}$$

where r, s, t are integers such that $r^2\, t = \deg(f)$ and s either vanishes or is a divisor of t. The cases $s = 0$ can be ruled out by the requirement that block diagonal period matrices are not allowed [47], being contributions from a particular boundary component of moduli space. The existence of the form (2.22) implies, among other things, that there exists a modular transformation such that the windings around the temperature direction of spacetime occur *only* around the single homology cycle a_1, with all other cycles being periodic. This means that the compactification conditions can be chosen to be

$$\begin{aligned} \oint\limits_{a_1'} \mathrm{d}x^0 &= \beta\, r + \sqrt{2}\,\pi\, R\,\mathrm{i}\, p_1' \ , \\ \oint\limits_{a_j'} \mathrm{d}x^0 &= \sqrt{2}\,\pi\, R\,\mathrm{i}\, p_j' \ , \quad j = 2, \dots, g \ , \\ \oint\limits_{b_i'} \mathrm{d}x^0 &= \sqrt{2}\,\pi\, R\,\mathrm{i}\, q_i' \ , \\ \oint\limits_{a_i'} \mathrm{d}x^9 &= \sqrt{2}\,\pi\, R\, p_i' \ , \end{aligned}$$

$$\oint_{b_i'} \mathrm{d}x^9 \quad = \quad \sqrt{2}\,\pi\,R\,q_i' \tag{2.23}$$

for $i = 1, \ldots, g$, with the transverse components $\boldsymbol{x}$ periodic around the new basis of homology cycles $\boldsymbol{a}', \boldsymbol{b}'$ of Σ_g. In addition, after summation over $\boldsymbol{p}', \boldsymbol{q}'$ only the homology cycles a_2 and b_1 wrap around the light-cone.

Reduction depends on the number theoretic properties of the entries of the integral matrix M and is explicitly carried out by using the $2g \times 2g$ symplectic matrices

$$\begin{pmatrix} 1\!\!1_g & S \\ \mathbf{0}_g & 1\!\!1_g \end{pmatrix} \quad , \quad \begin{pmatrix} 1\!\!1_g & \mathbf{0}_g \\ S & 1\!\!1_g \end{pmatrix} \quad , \quad \begin{pmatrix} A & \mathbf{0}_g \\ \mathbf{0}_g & \left(A^{-1}\right)^\top \end{pmatrix} \ , \tag{2.24}$$

where S is a symmetric $g \times g$ integral matrix and $A \in SL(g,\mathbb{Z})$. By regarding the matrix (2.9) as consisting of two $2 \times g$ block matrices $\mathsf{M} = (\mathsf{M}_1, \mathsf{M}_2)$, the matrices (2.24) interpolate between these blocks via elementary row and column operations. Using the normal form (2.21) one can transform (2.13) into the equation of the Weierstrass-Poincaré theorem

$$(1\!\!1_g, \Omega)\,\mathsf{T} = \mathsf{F} \begin{pmatrix} 1 & 0 & \ldots & 0 & -\frac{\sigma_1}{t\,\sigma_2} & \boldsymbol{q} \\ \mathbf{0}_{(g-1)\times 1} & & 1\!\!1_{g-1} & \boldsymbol{q}^\top & \mathsf{Z} & \end{pmatrix} \ , \tag{2.25}$$

where F is a non-singular $g \times g$ complex matrix, $\boldsymbol{q} = (-\frac{s}{t}, 0, \ldots, 0)$ is a $(g-1)$-vector, the complex numbers σ_1, σ_2 are defined by $(\sigma_1, \sigma_2) = (1, \mathrm{i}\,\nu)\,\mathsf{S}$, and Z is a $(g-1) \times (g-1)$ complex matrix satisfying the Riemann bilinear relations which can be found after explicit construction of the symplectic transformation. Because the vector $\boldsymbol{q}$ is rational-valued, the corresponding genus g theta-functions factorize into products of theta-functions of lower genera based on the curves with periods $\frac{\sigma_1}{t\,\sigma_2}$ and Z.

The reduction to the Poincaré normal form (2.22) can be thought of as a gauge fixing of the large diffeomorphism symmetry (the mapping class group) of the Riemann surface Σ_g. There is still then a residual gauge symmetry left over, which we will fix by restricting to those modular transformations which preserve the corresponding reduced compactification conditions. This defines a proper subgroup $\mathcal{G} \subset Sp(2g,\mathbb{Z})$, and so it will *extend* the fundamental modular region for the action of $Sp(2g,\mathbb{Z})$ on $\mathcal{H}_g$ from $\mathcal{F}_g$ to some domain $\mathcal{F}_g'$. Modular invariance is then restored via the observation [46] that the new region $\mathcal{F}_g'$ is composed of an infinite number of images of the fundamental domain $\mathcal{F}_g$ under certain elements of the modular group. The sum over all copies of $\mathcal{F}_g$ in $\mathcal{F}_g'$ may be implemented by a sum over all elements of the coset $Sp(2g,\mathbb{Z})/\mathcal{G}$. The corresponding constraints on the period matrix Ω in (2.25) reduce the complex dimension $3g-3$ of moduli space to $2g-3$. In addition, there is discrete data contained in the compactification integers, such as those arising from the requirement that the real-valued symmetric matrix Ω_2 be positive. This gives a partial discretization of the Riemann moduli space $\mathcal{M}_g$ to the Hurwitz moduli space of holomorphic maps, with the $2g-3$ moduli given by the branching data required to build the cover Σ_g from its base $\mathbb{T}^2_{\mathrm{i}\,\nu}$. The Hurwitz space can be embedded as an analytic subvariety of $\mathcal{M}_g$ [26, 27].

2.3 One-Loop Computation

It is instructive to recall the genus one situation [31, 33]. Then all covers $\Sigma_1 \to \mathbb{T}^2_{\mathrm{i}\,\nu}$ are unbranched. In this case one can deduce the period constraint of (2.6) by elementary methods

which exhibit the geometric construction of the covering torus from the base torus in terms of the compactification integers specified by (2.9). For this, let us regard the torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$ as the quotient of the complex plane $\mathbb{C}$ by a lattice $\Lambda = \langle e_1, e_2 \rangle := \mathbb{Z}\, e_1 \oplus \mathbb{Z}\, e_2$ of rank 2 generated by two-vectors e_1 and e_2. The isomorphism classes of unramified covers $\Sigma_1 \to \mathbb{T}^2_{\mathrm{i}\,\nu}$ of degree N then correspond to the inequivalent sublattices $\Lambda' \subset \Lambda$ of index $[\Lambda : \Lambda'] = N$. These may be found as follows. Let $f_1 = r'\, e_1 \in \Lambda'$ be the smallest multiple of e_1. Then there exists $f_2 = s'\, e_1 + m'\, e_2 \in \Lambda'$ with $s' < r'$ such that Λ' is generated by f_1 and f_2 over $\mathbb{Z}$. The index of this lattice is $r'\, m'$. As a consequence, for each integer r' dividing the index N there are r' inequivalent sub-lattices

$$\big\langle r'\, e_1, \tfrac{N}{r'}\, e_2 \big\rangle \quad , \quad \big\langle r'\, e_1, e_1 + \tfrac{N}{r'}\, e_2 \big\rangle \quad , \quad \dots \quad , \quad \big\langle r'\, e_1, (r'-1)\, e_1 + \tfrac{N}{r'}\, e_2 \big\rangle \ . \tag{2.26}$$

It follows that the number of inequivalent sublattices $\Lambda' \subset \Lambda$ of index $[\Lambda : \Lambda'] = N$ is

$$\sigma_1(N) = \sum_{r'|N} r' \ , \tag{2.27}$$

and the moduli of the corresponding covers are given by

$$\tau = \frac{s' + \frac{\mathrm{i}}{\nu}\, m'}{r'} \ . \tag{2.28}$$

We will now use the Weierstrass-Poincaré reduction to show that solving the reduced constraint in this case gives the same moduli (2.28) of the covers constructed from the base modular parameter $\mathrm{i}\,\nu$. The integers $n' = 0, m' \in \mathbb{Z}$ are defined by the $SL(2,\mathbb{Z})$ transformation

$$n' = 0 = D\, n + C\, m \qquad , \qquad -\, m' = B\, n + A\, m \ . \tag{2.29}$$

The first equation is solved by the relatively prime integers $C = -n/\gcd(n,m)$ and $D = m/\gcd(n,m)$. Now we use the fact that the set of integers $\mathbb{Z}$ is a principal ideal domain, which implies that there exists integers A and B such that

$$A\, m + B\, n = \gcd(n,m) \ . \tag{2.30}$$

Reduction for the genus one case is thus very simple, as all the windings of the cover Σ_1 around the temperature direction are put into the b cycle by the $SL(2,\mathbb{Z})$ transformation generated by the unimodular matrix

$$\mathsf{T}_1 = \begin{pmatrix} \frac{m}{\gcd(n,m)} & B \\ -\frac{n}{\gcd(n,m)} & A \end{pmatrix} \ . \tag{2.31}$$

Furthermore, from (2.29) it follows that the sole temperature winding integer is given by the greatest common divisor of the original two winding numbers as

$$m' = -\gcd(n,m) \ . \tag{2.32}$$

The constraint equation for the modulus τ of the cover Σ_1 is given by

$$\mathsf{H}^\top\, (1,\tau) = (1,\, \mathrm{i}\,\nu) \begin{pmatrix} n & m \\ r & s \end{pmatrix} = (1,\, \mathrm{i}\,\nu) \begin{pmatrix} 0 & -m' \\ r' & s' \end{pmatrix} \begin{pmatrix} A & -B \\ \frac{n}{\gcd(n,m)} & \frac{m}{\gcd(n,m)} \end{pmatrix} \ , \tag{2.33}$$

which can be solved explicitly to determine τ as in (2.28) with

$$r' = \frac{m\, r - n\, s}{\gcd(n,m)} \qquad , \qquad s' = B\, r + A\, s \ . \tag{2.34}$$

The genus one fundamental domain is given by

$$\Delta := \mathcal{F}_1 = \left\{ \tau \in \mathbb{C} \;\Big|\; -\tfrac{1}{2} < \tau_1 \leq \tfrac{1}{2} \;,\; |\tau|^2 \geq 1 \;,\; \tau_2 > 0 \right\} \;. \tag{2.35}$$

Requiring the reduced compactification constraints to be modular invariant sets $C = 0$ and $A = D = 1$ in (2.16). Thus only the translations $\tau \mapsto \tau + B$, $B \in \mathbb{Z}$ survive under the action of the restricted modular group $\mathcal{G}$ on Teichmüller space, and the fundamental modular region is extended to the strip

$$\Delta' := \mathcal{F}_1' = \left\{ \tau \in \mathbb{C} \;\Big|\; -\tfrac{1}{2} < \tau_1 \leq \tfrac{1}{2} \;,\; \tau_2 > 0 \right\} \;. \tag{2.36}$$

Requiring that $\tau \in \Delta'$ is then equivalent to $s' \in \mathbb{Z}/r'\,\mathbb{Z}$, $N := m'\,r' > 0$.

The integration measure on moduli space is obtained by computing the standard zero temperature, chiral Laplacian determinants on the torus induced by integrating out the ten worldsheet bosonic fields x^μ, their superpartners ψ^μ, and the ghosts, in a given spin structure. The GSO projection then dictates to sum over the three even spin structures in each of the left and right moving sectors of the worldsheet field theory (The odd spin structure $(1,1)$ yields a vanishing contribution due to the zero modes of the free worldsheet fermion fields ψ^μ). The appropriate modification which makes the spacetime fermion fields antiperiodic inserts an extra phase factor $(-1)^{m'}$ in front of the GSO phase associated with the $(0,1)$ spin structure. The modular invariant, finite-temperature superstring measure is thereby given as [5]

$$\mathrm{d}\mu_1^{(m')}(\tau\,,\,\overline{\tau}) = \left(\frac{1}{4\pi^2\,\alpha'}\right)^5 \frac{\mathrm{d}^2\tau}{(\tau_2)^6}\,\frac{1}{4\big|\eta(\tau)\big|^8}\,\Big|\theta_2(0|\tau)^4 - \theta_3(0|\tau)^4 + \mathrm{e}^{\,\pi\,\mathrm{i}\,m'}\,\theta_4(0|\tau)^4\Big|^2 \;. \tag{2.37}$$

Here the Jacobi-Erderlyi functions $\theta_a(z|\tau)$, $a = 2,3,4$ (which are induced by the spacetime fermion fields and the superconformal ghost fields) are defined in terms of the three even characteristic, genus one theta-functions as $\theta_2 = \theta\binom{1}{0}$, $\theta_3 = \theta\binom{0}{0}$, and $\theta_4 = \theta\binom{0}{1}$, where

$$\theta\begin{pmatrix} a \\ b \end{pmatrix}(z|\tau) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\,\pi\,\mathrm{i}\,(n+\frac{1}{2}\,a)^2\,\tau}\;\;\mathrm{e}^{\,2\pi\,\mathrm{i}\,(n+\frac{1}{2}\,a)\,(z+\frac{1}{2}\,b)} \tag{2.38}$$

are holomorphic functions of $(z|\tau) \in \mathbb{C} \times \mathcal{H}_1$ for $a, b \in \mathbb{R}$, while

$$\eta(\tau) = \tfrac{1}{2}\,\theta_2(0|\tau)\,\theta_3(0|\tau)\,\theta_4(0|\tau) \tag{2.39}$$

is the Dedekind function (which is induced by the spacetime boson fields and the conformal ghost fields). By using the Jacobi abstruse identity

$$\theta_3(0|\tau)^4 - \theta_4(0|\tau)^4 - \theta_2(0|\tau)^4 = 0 \;, \tag{2.40}$$

we can simplify the expression (2.37) to

$$\mathrm{d}\mu_1^{(m')}(\tau\,,\,\overline{\tau}) = \left(\frac{1}{4\pi^2\,\alpha'}\right)^5 \frac{\mathrm{d}^2\tau}{(\tau_2)^6}\,\frac{\left(1 - \mathrm{e}^{\,\pi\,\mathrm{i}\,m'}\right)\big|\theta_4(0|\tau)\big|^8}{2\big|\eta(\tau)\big|^8} \;. \tag{2.41}$$

By substituting all of these expressions back into the genus one free energy (2.6) and integrating the delta-function with the appropriate Jacobian factor, we arrive finally at

$$F_1 = -\frac{1}{\sqrt{2}\,\pi\,R\,\beta}\sum_{N=1}^{\infty} \mathrm{e}^{\,-\frac{\beta\,N}{\sqrt{2}\,R}}\;\mathbf{H}_N * \left[\left(\frac{1}{4\pi^2\,\alpha'\,\tau_2}\right)^4 \frac{\big|\theta_4(0|\tau)\big|^8}{\big|\eta(\tau)\big|^8}\right]\Bigg|_{\tau=\mathrm{i}\,/\nu} \;, \tag{2.42}$$

where $\mathbf{H}_N$ are the (restricted) Hecke operators [3] whose actions on a modular invariant function $f(\tau,\overline{\tau}\,)$ on Teichmüller space are defined by

$$\mathbf{H}_N * f(\tau,\overline{\tau}\,)=\frac{1}{N}\sum_{\substack{m'\,r'=N\\ m'\ \mathrm{odd}}}\ \sum_{s'\in\mathbb{Z}/r'\,\mathbb{Z}} f\left(\tfrac{s'+\tau\,m'}{r'},\tfrac{s'+\overline{\tau}\,m'}{r'}\right)\ . \tag{2.43}$$

By applying the modular transformation $\tau\mapsto-1/\tau$ and using the transformation rules

$$\theta_4(\tfrac{z}{\tau}|-\tfrac{1}{\tau})=\sqrt{-\,\mathrm{i}\,\tau}\ \ \mathrm{e}^{\pi\,\mathrm{i}\,z^2/\tau}\ \theta_2(z|\tau)\qquad,\qquad\eta(-\tfrac{1}{\tau})=\sqrt{-\,\mathrm{i}\,\tau}\ \eta(\tau)\ , \tag{2.44}$$

we can transform the expression (2.42) into the equivalent form

$$F_1=-\frac{1}{\sqrt{2}\,\pi\,R\,\beta}\sum_{N=1}^\infty\ \mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}\ \mathbf{H}_N*\left[\left(\frac{1}{4\pi^2\,\alpha'\,\tau_2}\right)^4\ \frac{\big|\theta_2(0|\tau)\big|^8}{\big|\eta(\tau)\big|^8}\right]\Bigg|_{\tau=\mathrm{i}\,\nu}\ . \tag{2.45}$$

The operand of the Hecke operators in (2.45) is the partition function of a first quantized Green-Schwarz superstring, so that the expression (2.45) has a natural interpretation as a map from a first quantized to a second quantized superstring theory [23]. The discrete Teichmüller parameters (2.28) indicate how the homology cycles of the base $\mathbb{T}^2_{\mathrm{i}\,\nu}$ wind around the cycles of the unbranched cover Σ_1. The combinatorics of enumerating unbranched covers of the torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$ are thereby elegantly accounted for by the Hecke operators acting on the partition function of a superconformal field theory, with toroidal worldsheet and target space $\mathbb{R}^8$, in (2.45). This result agrees with both the computation using operator quantization in light-cone gauge and in matrix string theory [31], as well as in the superconformal field theory on the symmetric product orbifold (1.3) [28]. The calculation can also be applied to bosonic and heterotic strings, with the final result always being the insertion of the appropriate one-loop light-cone Green-Schwarz string partition function in the operand of the Hecke operator in (2.45). In what follows we shall extend these one-loop calculations to the case of genus two branched covers Σ_2 of the torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$.

3 Bosonic Strings

We will now extend the calculation of Section 2.3 by computing the two-loop free energy F_2 in (2.6). As a warm up, in this section we will look at the simpler setting of bosonic string theory (whose action is obtained from (2.1) by dropping all Grassmann fields in 26 spacetime dimensions) for which the moduli space integration measure is more manageable. This will make the various reduction techniques that we present more transparent. They will also carry through to the superstring and heterotic string cases which will be studied in the next two sections. There is a fairly complete picture of Teichmüller space and moduli space at genus two. Every genus two surface admits a hyperelliptic representation as a double cover of the complex plane with three quadratic branch cuts supported by six branch points. While this description is useful for describing interacting matrix strings [58, 14], it is not the natural parametrization for DLCQ strings.

3.1 Two-Loop Worldsheet Contributions

The two-loop free energy is given by a sum over (equivalence classes of) non-constant holomorphic maps $f:\Sigma_2\to\mathbb{T}^2_{\mathrm{i}\,\nu}$. Let us begin by summarizing some useful facts about these

contributing worldsheets [17]. By the Riemann-Hurwitz theorem, the total branching number B for the branched cover of a torus by a genus two surface Σ_2 is $B = 2$. This means that a covering $f : \Sigma_2 \to \mathbb{T}^2_{\mathrm{i}\,\nu}$ has three possible types of singularities: (a) Two simple branch points; (b) one branch point with two ramification points each of ramification index 2; or (c) one branch point with one ramification point of ramification index 3. The singularity types (b) and (c) can each be thought of as degenerate limits of type (a), which in this sense represents the generic situation.

The lifting of curves from $\mathbb{T}^2_{\mathrm{i}\,\nu}$ to the covering space Σ_2 induces a group homomorphism

$$f_\# \ : \ \pi_1\big(\mathbb{T}^2_{\mathrm{i}\,\nu} \setminus \mathcal{B}_f\big) \ \longrightarrow \ S_N \tag{3.1}$$

where $\mathcal{B}_f \subset \mathbb{T}^2_{\mathrm{i}\,\nu}$ is the branch locus of the covering map f, $N = \deg(f)$, and $\pi_1(\mathbb{T}^2_{\mathrm{i}\,\nu} \setminus \mathcal{B}_f) \cong \langle \alpha, \beta, \gamma_1, \gamma_2 \mid \alpha\,\beta\,\alpha^{-1}\,\beta^{-1}\,\gamma_1\,\gamma_2 = \mathbb{1} \rangle$ (with $\gamma_2 = \mathbb{1}$ in the case that $\mathcal{B}_f$ consists of a single non-simple branch point). Let γ_t be a homotopy generator which surrounds a branch point $t \in \mathcal{B}_f \subset \mathbb{T}^2_{\mathrm{i}\,\nu}$. If t is simple, then the permutation $f_\#(\gamma_t) \in S_N$ has a single non-trivial cycle of length 2. Otherwise, $f_\#(\gamma_t)$ either contains two non-trivial cycles of length 2 or it has a single non-trivial cycle of length 3. Together with the canonical homology generators α, β, these permutations generate a transitive subgroup $\mathcal{T}_{N,\mathcal{B}_f}$ of S_N and the induced map (3.1) is an isomorphism onto this subgroup. There is a one-to-one correspondence between elements of $\mathcal{T}_{N,\mathcal{B}_f}$ and irreducible branched covers. The two-loop free energy that we obtain in this and the subsequent sections are thus generating functions for the orbits in $\mathcal{T}_{N,\mathcal{B}_f}$ under conjugation by permutations in S_N.

3.2 Modular Parameters

We will now find the most general form of the 2×2 period matrix Ω of the covering surface Σ_2. This will be achieved by using a modified version of the reduction technique described in Section 2.2 to solve the genus two constraint which gives the moduli of the genus two branched covers of the torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$. The constraint equation (2.13) in this case reads

$$\mathsf{H}^\top\,(\mathbb{1}_2, \Omega) = (1,\, \mathrm{i}\,\nu) \begin{pmatrix} n_1 & n_2 & m_1 & m_2 \\ r_1 & r_2 & s_1 & s_2 \end{pmatrix} , \tag{3.2}$$

where $\sum_{i=1,2} (n_i\, s_i - m_i\, r_i) = \deg(f) =: N$ is the degree of the cover.

As in the one-loop calculation, it is possible to calculate part of the matrix T appearing in the Frobenius normal form (2.21) by choosing integers A_i, B_i such that

$$A_i\, m_i + B_i\, n_i = \gcd(n_i, m_i) =: n_i' \ , \quad i = 1,2 \ . \tag{3.3}$$

Then the $Sp(4,\mathbb{Z})$ matrix

$$\Lambda_a = \begin{pmatrix} B_1 & 0 & -\frac{m_1}{n_1'} & 0 \\ 0 & B_2 & 0 & -\frac{m_2}{n_2'} \\ A_1 & 0 & \frac{n_1}{n_1'} & 0 \\ 0 & A_2 & 0 & \frac{n_2}{n_2'} \end{pmatrix} \tag{3.4}$$

transfers all windings from the b_i homology cycles to the a_i cycles, i.e. it defines a Rankin-Selberg modular transformation (2.17) for which the 2×4 integral matrix M becomes

$$\mathsf{M} \ \longmapsto \ \begin{pmatrix} n_1' & n_2' & 0 & 0 \\ r_1' & r_2' & s_1' & s_2' \end{pmatrix} \tag{3.5}$$

with $r_i' = B_i\, r_i + A_i\, s_i$ and $s_i' = \frac{1}{n_i'}\,(n_i\, s_i - m_i\, s_i)$ for $i = 1, 2$. The matrix Λ_a belongs to an $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ subgroup of the full modular group $PSp(4,\mathbb{Z}) \cong SO(3,2,\mathbb{Z})$.

The next step is to move the temperature windings from the a_2 cycle to the a_1 cycle. For this, we introduce two further integers U_1 and U_2 with

$$U_1\, n_1' + U_2\, n_2' = \gcd(n_1', n_2') =: r\ . \tag{3.6}$$

Then the $Sp(4,\mathbb{Z})$ matrix Λ_b given by

$$\Lambda_b = \begin{pmatrix} U_1 & -\frac{n_2'}{r} & 0 & 0 \\ U_2 & \frac{n_1'}{r} & 0 & 0 \\ 0 & 0 & \frac{n_1'}{r} & -U_2 \\ 0 & 0 & \frac{n_2'}{r} & U_1 \end{pmatrix} \tag{3.7}$$

will perform the necessary operation. It belongs to an $SL(2,\mathbb{Z})$ subgroup of the mapping class group. The desired transformation of M for which all temperature windings have been transfered to the a_1 homology cycle is therefore described by the matrix

$$\mathsf{M}' := \mathsf{M}\, \Lambda_a\, \Lambda_b = \begin{pmatrix} r & 0 & 0 & 0 \\ x' & y' & z' & w \end{pmatrix} \tag{3.8}$$

where $x' = U_1\, r_1' + U_2\, r_2'$, $y' = \frac{1}{r}\,(n_1'\, r_2' - n_2'\, r_1')$, $z' = \frac{1}{r}\,(n_1'\, s_1' + n_2'\, s_2')$ and $w = U_1\, s_2' - U_2\, s_1'$.

We now construct a third transformation matrix $\Lambda_c \in Sp(4,\mathbb{Z})$ by disregarding the first and third columns of the matrix (3.8) and writing

$$\begin{pmatrix} 0 & 0 \\ y' & w \end{pmatrix} \begin{pmatrix} Y & -\frac{w}{z} \\ W & \frac{y'}{z} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}\ , \tag{3.9}$$

where the integers Y and W obey

$$Y\, y' + W\, w = \gcd(y', w) =: z\ . \tag{3.10}$$

This does not affect the zeroes in the first row of (3.8), and the symplectic matrix $\mathsf{T} = \Lambda_a\, \Lambda_b\, \Lambda_c$ finally reduces the matrix M to the form

$$\mathsf{M}' \longmapsto \begin{pmatrix} r & 0 & 0 & 0 \\ x & y & z & 0 \end{pmatrix} \tag{3.11}$$

with $x, y, z \in \mathbb{Z}$. Note that we do not apply the 2×2 matrix S here, which affects an $SL(2,\mathbb{Z})$ modular transformation of the base $\mathbb{T}^2_{\mathrm{i}\,\nu}$. The complete Poincaré normal form (2.22) is derived in Appendix A.

In this way the constraint equation (3.2) reduces to

$$\mathsf{H}^\top\, (\mathbb{1}_2, \Omega) = (1, \mathrm{i}\,\nu) \begin{pmatrix} r & 0 & 0 & 0 \\ x & y & z & 0 \end{pmatrix} \mathsf{T}\ . \tag{3.12}$$

Now we factor out a symplectic unit on the right-hand side of this equation in order that the eventual solution of the constraint equation gives a period matrix with rational-valued off-diagonal elements. This gives

$$\mathsf{H}^\top\, (\mathbb{1}_2, \Omega) = (1, \mathrm{i}\,\nu) \begin{pmatrix} 0 & 0 & -r & 0 \\ z & 0 & -x & -y \end{pmatrix} \mathsf{J}_2\, \mathsf{T}\ . \tag{3.13}$$

The matrix $\mathsf{J}_2\,\mathsf{T} \in Sp(4,\mathbb{Z})$ is non-singular, and its inverse $(\mathsf{J}_2\,\mathsf{T})^{-1}$ acts on the left-hand side of (3.13) as a modular transformation of the period matrix Ω and the pullback vector $\mathsf{H}^\top$. Parametrizing it by a block matrix of the form (2.16), one has

$$\mathsf{H}^\top\,(1\!\!1_2,\Omega)\,(\mathsf{J}_2\,\mathsf{T})^{-1} = \mathsf{H}^\top(C\,\Omega+D)\,\Big(1\!\!1_2\,,\,(C\,\Omega+D)^{-1}\,(A\,\Omega+B)\Big) =: \mathsf{H}'\,\big(1\!\!1_2,\Omega'\big) \tag{3.14}$$

giving

$$\mathsf{H}'\,\big(1\!\!1_2,\Omega'\big) = (1,\,\mathrm{i}\,\nu)\,\begin{pmatrix} 0 & 0 & -r & 0 \\ z & 0 & -x & -y \end{pmatrix}\,. \tag{3.15}$$

We can now solve the constraint (3.15) to get

$$\mathsf{H} = (1,\,\mathrm{i}\,\nu)\,\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} = (\,\mathrm{i}\,\nu\,z,0) \tag{3.16}$$

and

$$\mathsf{H}\,\Omega = (\,\mathrm{i}\,\nu\,z,0)\,\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{pmatrix} = (1,\,\mathrm{i}\,\nu)\,\begin{pmatrix} -r & 0 \\ -x & -y \end{pmatrix}\,, \tag{3.17}$$

where for notational ease we have dropped the primes indicating the modular transformations (The free energy is modular invariant). The period matrix is finally given in the form

$$\Omega = \begin{pmatrix} -\frac{x+\frac{r}{\mathrm{i}\,\nu}}{z} & -\frac{y}{z} \\ -\frac{y}{z} & \Omega_{22} \end{pmatrix} \tag{3.18}$$

with $r,x,y,z \in \mathbb{Z}$ and $\Omega_{22} \in \mathcal{H}_1$. This form of the period matrix has a natural geometrical interpretation. The diagonal elements are related to the moduli of two tori which have been sewn together along the branch cut of $\mathbb{T}^2_{\mathrm{i}\,\nu}$ to form the genus two cover. The element $-\frac{x+r/\mathrm{i}\,\nu}{z}$ is the modulus of a degree $N = r\,z$ unbranched cover Σ_1 of the torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$ as obtained in Section 2.3. The off-diagonal element is a measure of the radius and length of the cylinder joining the two tori when they are glued together along the branch cut of $\mathbb{T}^2_{\mathrm{i}\,\nu}$. This picture will be elucidated later on when we study degeneration limits of the branched covers Σ_2 in Section 6. Using the projective modular symmetry $PSp(4,\mathbb{Z})$ defining the moduli space $\mathcal{M}_2$, we will identify $\Omega \sim -\Omega$ in (3.18).

This calculation demonstrates that the existence of the covering $f:\Sigma_2 \to \mathbb{T}^2_{\mathrm{i}\,\nu}$, reducing a holomorphic differential on Σ_2 to an elliptic one (2.11), necessarily implies [11] the existence of another (generally distinct) covering $f':\Sigma_2 \to \mathbb{T}^2_\tau$ which leads to a reduction of some other independent holomorphic differential on Σ_2 to an elliptic one. In this case, the Jacobian of the curve Σ_2 represents a fibration whose base and fibre are the elliptic curves $\mathbb{T}^2_{\mathrm{i}\,\nu}$ and $\mathbb{T}^2_\tau$, with the commutative diagram

$$\begin{tikzcd} & \mathbb{T}^2_\tau & \\ \Sigma_2 \arrow[ur, "f'"] \arrow[r, "\mathfrak{A}"] \arrow[dr, "f"'] & \mathrm{Jac}(\Sigma_2) \arrow[u, "\Psi'"'] \arrow[d, "\Psi"] \; . & \\ & \mathbb{T}^2_{\mathrm{i}\,\nu} & \end{tikzcd} \tag{3.19}$$

The curve Σ_2 is embedded by the Abel map $\mathfrak{A}$ into its Jacobian variety as a divisor. The relationship (3.19) will then split the contribution to the two-loop effective string action from Σ_2 into individual contributions from the two tori $\mathbb{T}^2_\tau$ and $\mathbb{T}^2_{\mathrm{i}\,\nu}$, as we shall see explicitly below.

3.3 Moduli Space

The subgroup $\mathcal{G}$ of $Sp(4,\mathbb{Z})$ transformations which leave the structure of the integral matrices

$$\begin{pmatrix} 0 & 0 & -r & 0 \\ z & 0 & -x & -y \end{pmatrix} \tag{3.20}$$

in (3.15) invariant has four generators and consists of unimodular matrices of the generic form

$$\begin{pmatrix} 1 & A_{12} & B_{11} & B_{12} \\ 0 & A_{22} & B_{12} & B_{22} \\ 0 & 0 & 1 & 0 \\ 0 & C_{22} & D_{21} & D_{22} \end{pmatrix} \tag{3.21}$$

which obey the non-linear constraints

$$\begin{aligned} A_{22}\, D_{22} - B_{22}\, C_{22} &= 1 \ , \\ A_{22}\, D_{21} - B_{21}\, C_{22} &= A_{12} \ , \\ B_{21}\, D_{22} - B_{22}\, D_{21} &= B_{12} \ . \end{aligned} \tag{3.22}$$

We choose B_{11} and B_{12} as arbitrary integers. From the Hopf condition (2.14) it follows that the subgroup $\mathcal{G}$ preserves the two integers r and z. Under a modular transformation by an element (3.21) of the group $\mathcal{G}$ the period matrix transforms according to (2.19). By using (3.22) one finds that the matrix elements of Ω have the transformation properties

$$\begin{aligned} \Omega_{11} &\longmapsto \Omega_{11} + B_{11} - \frac{C_{22}\,(\Omega_{12})^2 + 2\, D_{21}\, \Omega_{12} + B_{12}\, D_{21}}{C_{22}\, \Omega_{22} + D_{22}} \ , \\ \Omega_{12} &\longmapsto A_{22}\, \Omega_{12} + B_{21} - \frac{A_{22}\, \Omega_{22} + B_{22}}{C_{22}\, \Omega_{22} + D_{22}} \left(C_{22}\, \Omega_{12} + D_{21}\right) \ , \\ \Omega_{22} &\longmapsto \frac{A_{22}\, \Omega_{22} + B_{22}}{C_{22}\, \Omega_{22} + D_{22}} \ . \end{aligned} \tag{3.23}$$

Note that Ω_{22} transforms under a genus one $SL(2,\mathbb{Z})$ modular transformation. In addition the positivity of Ω_2 yields the quadratic constraints

$$\mathrm{Im}(\Omega_{11}) > 0 \ , \quad \mathrm{Im}(\Omega_{22}) > 0 \ , \quad (\mathrm{Im}\, \Omega_{12})^2 < \mathrm{Im}(\Omega_{11})\ \mathrm{Im}(\Omega_{22}) \ . \tag{3.24}$$

From (3.23) and (3.24) it follows that the period matrix take values in the extended fundamental domain

$$\mathcal{F}_2' = \Big\{ \Omega \in \mathcal{H}_2 \ \Big| \ \Omega_{11} \in \Delta' \ , \ \Omega_{22} \in \Delta \ , \ \Omega_{12} \in \mathcal{P}_{\Omega_{22}} \Big\} \tag{3.25}$$

written in terms of the elliptic fundamental domains (2.35) and (2.36) along with the parallelogram

$$\mathcal{P}_\tau = \Big\{ \sigma_1 + \tau\, \sigma_2 \ \Big| \ \sigma_1, \sigma_2 \in [0,1] \Big\} \tag{3.26}$$

in the upper complex half-plane. The domain (3.25) is the same as the modular region obtained using the ordinary Rankin-Selberg reduction [47]. This provides a complete picture of the moduli space of branched covers of a torus at genus two. The map which sends a Riemann surface Σ_2 to the equivalence class of the period matrix $\Omega \in \mathcal{M}_2$ is an isomorphism onto the subspace $\mathcal{M}_2 \setminus [\mathcal{H}_1 \times \mathcal{H}_1]$, where $[\mathcal{H}_1 \times \mathcal{H}_1]$ is the modular orbit of the space of diagonal period

matrices in $\mathcal{H}_2$ corresponding to the boundary component of moduli space where the surface Σ_2 degenerates into two tori. The general task of finding an explicit set of inequalities on the matrix elements of Ω which characterizes the corresponding fundamental modular domain $\mathcal{F}_2$ is a difficult highly non-linear mathematical problem. Here an explicit representation of moduli space has been obtained by using reduction and unfolding techniques. This is the main motivation behind our modification of the Poincaré normal form, as it leads to a much simpler and tractable fundamental modular region. For completeness, the complete moduli space corresponding to the fully reduced Poincaré normal form (2.22) at genus two is worked out explicitly in Appendix A.

The sums over the integers in (2.6) which characterize the branched covers are restricted by the requirement that they count only the moduli (3.18) lying in the extended fundamental domain (3.25). The positivity constraint (3.24) and the Hopf condition (2.14) for the degree N of the covering map require $r,z\in\mathbb{N}$ such that $r\,z=N$. The two equivalence relations $\Omega_{12}\sim\Omega_{12}+1$ and $\Omega_{11}\sim\Omega_{11}+1$ imply that $x,y\in\mathbb{Z}/z\,\mathbb{Z}$, with $y\neq 0$ in order for the period matrix in $\mathcal{M}_2\setminus[\mathcal{H}_1\times\mathcal{H}_1]$ to correspond to a genus two curve Σ_2. The arbitrary complex number $\tau:=-\Omega_{22}\in\mathcal{H}_1$ is integrated over the genus one fundamental domain Δ. These ranges are all defined so that the modular orbit of diagonal period matrices is removed from $\mathcal{H}_2$.

3.4 Theta-Constants

The genus two theta-function with characteristics $\Theta:\mathrm{Jac}(\Sigma_2)\times\mathcal{H}_2\to\mathbb{C}$ is defined as the Fourier series [51]

$$\Theta\binom{\boldsymbol{a}}{\boldsymbol{b}}(\boldsymbol{z}|\Omega)=\sum_{\boldsymbol{n}\in\mathbb{Z}^2}\exp\left[\pi\,\mathrm{i}\,\left(\boldsymbol{n}+\tfrac12\,\boldsymbol{a}\right)\cdot\Omega\left(\boldsymbol{n}+\tfrac12\,\boldsymbol{a}\right)+2\pi\,\mathrm{i}\,\left(\boldsymbol{n}+\tfrac12\,\boldsymbol{a}\right)\cdot\left(\boldsymbol{z}+\tfrac12\,\boldsymbol{b}\right)\right]\ . \tag{3.27}$$

It is a holomorphic function of $(\boldsymbol{z}|\Omega)\in\mathbb{C}^2\times\mathcal{H}_2$ for the characteristics $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^2$. For $\boldsymbol{a},\boldsymbol{b}\in\{0,1\}\times\{0,1\}$ the theta-function is even if $\boldsymbol{a}\cdot\boldsymbol{b}\equiv 0$ mod 2, odd if $\boldsymbol{a}\cdot\boldsymbol{b}\equiv 1$ mod 2. There are ten even genus two theta-functions and six odd ones. We can write (3.27) in a form without characteristics by factorizing a phase to get

$$\Theta\binom{\boldsymbol{a}}{\boldsymbol{b}}(\boldsymbol{z}|\Omega)=\,\mathrm{e}^{\,\frac{\pi\,\mathrm{i}}{4}\,\boldsymbol{a}\cdot\Omega\boldsymbol{a}+\pi\,\mathrm{i}\,\boldsymbol{a}\cdot(\boldsymbol{z}+\frac12\,\boldsymbol{b})}\ \Theta\binom{\boldsymbol{0}}{\boldsymbol{0}}\ \left(\boldsymbol{z}+\tfrac12\,\Omega\boldsymbol{a}+\tfrac12\,\boldsymbol{b}\,\middle|\,\Omega\right) \tag{3.28}$$

We can now use the reduction (3.18) to decompose the genus two theta-function (3.27) in terms of genus one theta-functions [45]. The exponent of the theta-function with zero characteristic in (3.28) is given by the quantity

$$\begin{aligned}\boldsymbol{k}\cdot\Omega\boldsymbol{k}+2\,\boldsymbol{k}\cdot\left(\boldsymbol{z}+\tfrac12\,\Omega\boldsymbol{a}+\tfrac12\,\boldsymbol{b}\right)\ &=\ (k_1)^2\,\Omega_{11}+2\,k_1\,\left(z_1+k_2\,\Omega_{12}+\tfrac12\,\Omega_{11}\,a_1+\tfrac12\,\Omega_{12}\,a_2+\tfrac12\,b_1\right)\\ &\quad+(k_2)^2\,\Omega_{22}+2\,k_2\,\left(z_2+\tfrac12\,\Omega_{12}\,a_1+\tfrac12\,\Omega_{22}\,a_2+\tfrac12\,b_2\right)\end{aligned} \tag{3.29}$$

for $\boldsymbol{k}=(k_1,k_2)\in\mathbb{Z}^2$. In the present case the period matrix (3.18) (after projective $\mathbb{Z}_2$ reflection) has rational-valued off-diagonal entries $\Omega_{12}=\frac{y}{z}=\frac{r\,y}{N}$. Let $k_2=n+N\,\tilde{k}_2$ where $\tilde{k}_2\in\mathbb{Z}$ and $0\le n\le N-1$. We may then rewrite (3.29) in the form

$$\begin{aligned}\boldsymbol{k}\cdot\Omega\boldsymbol{k}+2\,\boldsymbol{k}\cdot\left(\boldsymbol{z}+\tfrac12\,\Omega\boldsymbol{a}+\tfrac12\,\boldsymbol{b}\right)\ &=\ (k_1)^2\,\Omega_{11}+2\,k_1\,\left[z_1+\left(n+\tfrac{a_2}{2}\right)\,\Omega_{12}+\Omega_{11}\,\tfrac{a_1}{2}+\tfrac{b_1}{2}\right]\\ &\quad+2\,N\,\tilde{k}_2\,\Omega_{12}+\left(\tfrac{n}{N}+\tilde{k}_2\right)^2\,N^2\,\Omega_{22}\end{aligned}$$

$$+\,2\,N\,\left(\tfrac{n}{N}+\tilde{k}_2\right)\,\left(z_2+\Omega_{12}\,\tfrac{a_1}{2}+\Omega_{22}\,\tfrac{a_2}{2}+\tfrac{b_2}{2}\right)\ . \tag{3.30}$$

Once this expression is multiplied by $\pi\,\mathrm{i}$ and exponentiated, the term $2\pi\,\mathrm{i}\,N\,\tilde{k}_2\,\Omega_{12}$ can be dropped since it is an integer multiple of $2\pi\,\mathrm{i}$. In this way the genus two theta-function factorizes into elliptic theta-functions (2.38) as

$$\begin{aligned}\Theta\binom{\boldsymbol{a}}{\boldsymbol{b}}(\boldsymbol{z}|\Omega) \;&=\; \mathrm{e}^{-\pi\,\mathrm{i}\,a_1\,a_2\,r\,y/2\,N}\,\sum_{n=0}^{N-1}\,\mathrm{e}^{-\pi\,\mathrm{i}\,a_1\,n\,r\,y/N}\,\theta\binom{a_1}{b_1}\left(z_1+\left(n+\tfrac{a_2}{2}\right)\tfrac{r\,y}{N}\,\Big|\,\tfrac{r\,x+\frac{r^2}{\mathrm{i}\,\nu}}{N}\right)\\&\qquad\times\,\theta\binom{\frac{2\,n+a_2}{N}}{0}\left(N\,\left(z_2+\tfrac{a_1\,r\,y}{2\,N}+\tfrac{b_2}{2}\right)\Big|\,N^2\,\tau\right)\ .\end{aligned} \tag{3.31}$$

Each term in the sum over n in (3.31) contains a pair of theta-functions, one for each of the tori in (3.19).

Let us now restrict to theta-constants by setting $\boldsymbol{z}=\boldsymbol{0}$ and to integer characteristics $\boldsymbol{a},\boldsymbol{b}\in\{0,1\}\times\{0,1\}$. The decomposition (3.31) into genus one theta-functions can then be simplified somewhat by applying a Poisson resummation to the second set of theta-functions with fractional characteristics. For those characteristics, this results in the modular transformation property [9]

$$\theta\binom{a}{b}(z|\tau)=\frac{\mathrm{e}^{-\pi\,\mathrm{i}\,\left(\frac{z^2}{\tau}+\frac{a\,b}{2}\right)}}{\sqrt{-\,\mathrm{i}\,\tau}}\;\theta\binom{b}{0}\left(-\tfrac{a}{2}-\tfrac{z}{\tau}\,\Big|-\tfrac{1}{\tau}\right)\ . \tag{3.32}$$

Then the elliptic theta-functions are all given by standard integer characteristic Jacobi-Erderlyi functions θ_a for $a=1,2,3,4$ and (3.31) becomes

$$\Theta\binom{\boldsymbol{a}}{\boldsymbol{b}}(\boldsymbol{0}|\Omega)=\frac{\mathrm{e}^{\,\pi\,\mathrm{i}\,a_2\,b_2/2}}{N\,\sqrt{-\,\mathrm{i}\,\tau}}\,\sum_{n=0}^{N-1}(-1)^{b_2\,n}\,\theta\binom{a_1}{b_1}\left(\left(n+\tfrac{a_2}{2}\right)\tfrac{r\,y}{N}\,\Big|\,\tfrac{r\,x+\frac{r^2}{\mathrm{i}\,\nu}}{N}\right)\,\theta_\gamma\left(\tfrac{n+\frac{a_2}{2}}{N}\,\Big|-\tfrac{1}{N^2\,\tau}\right)\ , \tag{3.33}$$

where $\gamma=2$ (resp. $\gamma=3$) when the degree N and the connecting integer y are even (resp. odd).

3.5 Free Energy

We are finally ready to write down the genus two contribution to the bosonic free energy. In the critical bosonic string theory, the corresponding integration measure $\mathrm{d}\mu_2^{\rm bos}(\Omega,\overline{\Omega}\,)$ on moduli space is completely characterized by the fact that it is expressed in terms of the square of a holomorphic volume form on $\mathcal{M}_2$ [10, 49, 47], and by the requirement that it be free of global gravitational anomalies. It is independent of temperature and has no zeroes or singularities in the interior of moduli space, while on the boundary of moduli space (the $Sp(4,\mathbb{Z})$ orbit of $\mathcal{H}_1\times\mathcal{H}_1$ and infinity in $\mathcal{H}_2$) it has a second order pole. This uniquely determines the bosonic integration measure as an expansion in terms of modular forms. Using holomorphic factorization, it can thereby be shown [10, 49, 47, 19] that the modular invariant, bosonic genus two moduli space measure is given by

$$\mathrm{d}\mu_2^{\rm bos}(\Omega\,,\,\overline{\Omega}\,)=\left(\frac{1}{4\pi^2\,\alpha'}\right)^{12}\,\mathrm{d}^2\Omega_{11}\wedge\mathrm{d}^2\Omega_{22}\wedge\mathrm{d}^2\Omega_{12}\,\left(\det\Omega_2\right)^{-13}\,\Big|\Psi_{10}(\Omega)\Big|^{-2}\ , \tag{3.34}$$

where Ψ_{10} is the unique, parabolic modular form of weight ten which vanishes on the diagonal period matrices of $\mathcal{H}_2$ (This generalizes the Dedekind function (2.39) which comprises the one-loop moduli space density for bosonic strings [47]). It can be expressed in terms of the ten even

integer characteristic, genus two theta-constants as the holomorphic Siegel cusp form

$$\Psi_{10}(\Omega) = 2^{-12} \prod_{\boldsymbol{a}\cdot\boldsymbol{b}\equiv 0 \bmod 2} \left[\Theta\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}(\boldsymbol{0}\,|\,\Omega)\right]^2 . \tag{3.35}$$

After integrating the delta-function in (2.6) with the necessary Jacobian from (3.17) and (3.18), the bosonic free energy is thereby found to be

$$\begin{aligned} F_2^{\rm bos} &= -g_s^2 \left(\frac{1}{2\sqrt{2}\,\pi\,\beta\,R}\right)^{12} \sum_{N=1}^{\infty} \frac{\mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}}{N^2} \sum_{r\,z=N} \left(\frac{z}{r}\right)^{10} \\ &\quad \times \sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z}\\ y\neq 0}} \int_\Delta \frac{\mathrm{d}^2\tau}{(\tau_2)^{12}} \prod_{\boldsymbol{a}\cdot\boldsymbol{b}\equiv 0 \bmod 2} \left|\Theta\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}(\boldsymbol{0}|\Omega)\right|^{-4} . \end{aligned} \tag{3.36}$$

Using (3.33) the product over genus two theta-functions in (3.36) can be expressed in terms of a long string of integer characteristic Jacobi-Erderlyi functions θ_a, $a = 1,2,3,4$. Generically these are *not* theta-constants of the elliptic curves $\mathbb{T}^2_{\mathrm{i}\,\nu}$ and $\mathbb{T}^2_\tau$, as the connecting integers $y \in \mathbb{Z}/z\,\mathbb{Z}$ gluing the two tori together appear in their arguments. The sums over the remaining integers N, r, z, x give the summation over worldsheet instanton sectors $\Sigma_1 \to \mathbb{T}^2_{\mathrm{i}\,\nu}$ characterizing the Hecke algebra. The τ-integral in (3.36) gives the integration over the location of the branch cut on the base $\mathbb{T}^2_{\mathrm{i}\,\nu}$ which is used to construct the covering surface Σ_2 by gluing. This identification can be established by using Thomae formulas to express the branch points of the genus two curve transcendentally in terms of the theta-constants (3.33) [11]. We will return to these features in Section 6.

As an aside, it is interesting to note that the cusp form (3.35) also arises in the computation [34] of the elliptic genus of the Kummer surface K3 as the one-loop free energy of a single string given by

$$\chi_{\rm K3}(\zeta|\tau) = 8\sum_{a=2}^{4} \frac{\theta_a(\zeta|\tau)^2}{\theta_a(0|\tau)^2} , \tag{3.37}$$

where $\Omega_{12} = \zeta$ is the complexified Kähler form of the elliptic curve $\mathbb{T}^2_\tau$. The completion of the corresponding string partition function on the symmetric product orbifold of K3 to an automorphic form for the group $SO(3,2,\mathbb{Z})$ is simply (3.35). This form can also be interpreted as the denominator of a generalized Kac-Moody algebra [34, 41]. Our reduction formulas here and in what follows bear a remarkable similarity to this construction, with the K3 surface regarded as the resolution of the orbifold $(\mathbb{T}^2_{\mathrm{i}\,\nu} \times \mathbb{T}^2_\tau)/\mathbb{Z}_2$. It would be interesting to further pursue whether or not our two-loop partition functions admit deeper interpretations along these lines.

4 Superstrings

We now turn to our main object of interest, the two-loop superstring free energy. There are two new ingredients in this case that one must add to the calculation of the previous section. In the genus one case, the simplicity of the measure (2.37) is a result of the local cancellation between the longitudinal X and ghost B,C determinants on moduli space. This ceases to happen for genera $g > 1$, and in this case the calculations are notoriously subtle. We shall

take the standard prescription for obtaining the measure $\mathrm{d}\mu_2[{\boldsymbol{n} \atop \boldsymbol{m}}](\Omega,\overline{\Omega}\,)$ by integrating over the fermionic moduli [56]. The non-splitness of super-moduli space does not generically allow for a global, unambiguous reduction to ordinary moduli, since the Grassmann integrations lead to spurious gauge dependences in the form of total derivative terms on $\mathcal{M}_2$ [6]. The problem can be overcome by descending from super-moduli space to moduli space by projecting super-geometries onto super-period matrices [19]. The integration over Grassmann odd supermoduli is then performed by integrating over the fibers of this projection. With this, one can find a good global holomorphic gauge slice in Teichmüller space without spurious gauge dependences that could otherwise lead to modular anomalies in the measure on moduli space. For each even spin structure at $g=2$, slice-independence allows an arbitrary choice of worldsheet gravitino field insertion point [25] and the split gauge choice leads to an expression for the chiral superstring measure in terms of modular forms [19]. The contributions from odd spin structures again vanish as a result of the integration over fermionic zero modes. For fixed spin structure, the chiral measure allows for a unique modular covariant GSO projection [19], which must be appropriately modified [5] due to the finite-temperature supersymmetry breaking effects analogously to the one-loop case.

4.1 Spin Structures

Let us begin by setting some useful shorthand notations. A reduced genus two integer characteristic is a pair of vectors $\binom{\boldsymbol{a}}{\boldsymbol{b}}=\left({a_1 \atop a_2}\,{b_1 \atop b_2}\right)$ where each a_i, b_i, $i=1,2$ are either 0 or 1. A characteristic is even if $\boldsymbol{a}\cdot\boldsymbol{b}\equiv 0 \bmod 2$, odd if $\boldsymbol{a}\cdot\boldsymbol{b}\equiv 1 \bmod 2$. There are ten even characteristics and six odd characteristics associated to the distinct choices of spin structures on the genus two Riemann surface Σ_2, i.e. to the choices of a square root of the canonical line bundle over Σ_2. The ten even characteristics (spin structures) are denoted

$$\begin{array}{lllll} \delta_1=\binom{00}{00} & \delta_2=\binom{00}{01} & \delta_3=\binom{01}{00} & \delta_4=\binom{01}{01} & \delta_5=\binom{00}{10} \\ \delta_6=\binom{01}{10} & \delta_7=\binom{10}{00} & \delta_8=\binom{10}{01} & \delta_9=\binom{10}{10} & \delta_0=\binom{11}{11}\ . \end{array} \tag{4.1}$$

The odd spin structures are denoted

$$\nu_1=\binom{00}{11} \qquad \nu_2=\binom{01}{11} \qquad \nu_3=\binom{11}{00} \qquad \nu_4=\binom{11}{01} \qquad \nu_5=\binom{10}{11} \qquad \nu_6=\binom{11}{10}\ . \tag{4.2}$$

Integer characteristics may be summed modulo 2, componentwise as if they were 2×2 matrices. For example, $\nu_1+\nu_4+\nu_6=\delta_0$ and $\nu_2+\nu_3+\nu_5=\delta_0$. There is a two-to-one map from triples of odd characteristics which are pairwise distinct onto even characteristics. The relative signature between any two spin structures is defined by

$$\big\langle \tbinom{\boldsymbol{a}}{\boldsymbol{b}} \big| \tbinom{\boldsymbol{a}'}{\boldsymbol{b}'} \big\rangle = \exp\Big[\pi\,\mathrm{i}\,\big(\boldsymbol{a}\cdot\boldsymbol{b}'-\boldsymbol{b}\cdot\boldsymbol{a}'\,\big)\Big]\ . \tag{4.3}$$

4.2 GSO Projection

In order for spacetime fermions and spacetime bosons to have the correct statistics at finite temperature, the fermions must have antiperiodic boundary conditions and the bosons must be periodic around the temperature direction of the target space. The standard GSO projection is thus modified by phases which take into account the winding numbers $\boldsymbol{n}$ and $\boldsymbol{m}$ [5]. A genus two Riemann surface has 16 spin structures given by the generators (4.1) and (4.2) of the

cohomology group $H^1(\Sigma_2,\mathbb{Z}/2\,\mathbb{Z})=(\mathbb{Z}/2\,\mathbb{Z})^4$ which are in one-to-one correspondence with flat real line bundles $L\to\Sigma_2$. Define $\phi(L)=+1$ if the spin structure L is even and $\phi(L)=-1$ if it is odd. This quantity coincides with the mod 2 index [51]

$$\phi(L)=\exp\Big[\pi\,\mathrm{i}\,\dim H^0\big(\Sigma_2\,,\,Spin(\Sigma_2)\otimes L\big)\Big] \tag{4.4}$$

which counts the number of holomorphic sections of the twisted spinor bundle $Spin(\Sigma_2)\otimes L$ modulo 2. The reduction modulo 2 of $(\boldsymbol{n},\boldsymbol{m})$ defines the characteristic class in $H^1(\Sigma_2,\mathbb{Z}/2\,\mathbb{Z})$ of a flat connection of a real line bundle $\mathcal{L}_{(\boldsymbol{n},\boldsymbol{m})}\to\Sigma_2$ such that a holomorphic section of $\mathcal{L}_{(\boldsymbol{n},\boldsymbol{m})}$ changes by a phase $(-1)^{n_i}$ as one goes once around the a_i homology cycles and by $(-1)^{m_i}$ as one goes once around the b_i homology cycles. Given a spin structure L, the tensor product $L\otimes\mathcal{L}_{(\boldsymbol{n},\boldsymbol{m})}$ is another spin structure for any $\boldsymbol{n},\boldsymbol{m}$ and we define

$$U_L(\boldsymbol{n},\boldsymbol{m})=\phi(L)\,\phi(L\otimes\mathcal{L}_{(\boldsymbol{n},\boldsymbol{m})})\ . \tag{4.5}$$

The quantity (4.5) takes values $\pm\,1$, and it is the correct phase to insert into the sum over spin structures L and winding numbers $\boldsymbol{n},\boldsymbol{m}$ [5]. It is compatible with both factorization of Σ_2 to lower genus and modular invariance.

As an example, let us calculate $U_{\delta_5}(\boldsymbol{n},\boldsymbol{m})=U_{\left(\substack{00\\10}\right)}(\boldsymbol{n},\boldsymbol{m})$. The spin structure δ_5 is even so $U_{\delta_5}(\boldsymbol{n},\boldsymbol{m})=\phi(L_{\delta_5}\otimes\mathcal{L}_{(\boldsymbol{n},\boldsymbol{m})})$. We first calculate $U_{\delta_5}(n_1,0,0,0)$. For $n_1\in\mathbb{Z}$ odd one has $L_{\delta_5}\otimes\mathcal{L}_{(n_1,0,0,0)}=L_{\delta_9}$ and so $U_{\delta_5}(n_1,0,0,0)=\phi(L_{\delta_9})=1$. The spin structure $L_{\delta_5}\otimes\mathcal{L}$ is likewise even if $\mathcal{L}$ corresponds to wrapping the a_2 and b_1 cycles around the temperature direction odd numbers of times n_2 and m_1, and so we have

$$U_{\delta_5}(n_1,0,0,0)=U_{\delta_5}(0,0,m_1,0)=U_{\delta_5}(0,n_2,0,0)=1\ . \tag{4.6}$$

When the b_2 cycle wraps around the temperature direction an odd number of times m_2 one gets the spin structure $L_{\delta_5}\otimes\mathcal{L}_{(0,0,0,m_2)}=L_{\nu_1}$. The phase is then

$$U_{\delta_5}(0,0,0,m_2)=(-1)^{m_2}\ . \tag{4.7}$$

Taking into account pairs of cycles produces the phase factors

$$\begin{aligned}
U_{\delta_5}(0,n_2,m_1,0) &= 1\ ,\\
U_{\delta_5}(0,0,m_1,m_2) &= U_{\delta_5}(n_1,0,0,m_2)\ =\ (-1)^{m_2}\ ,\\
U_{\delta_5}(n_1,0,m_1,0) &= \tfrac{1}{2}\,\big(1+(-1)^{n_1}+(-1)^{m_1}-(-1)^{n_1+m_1}\big)\ ,\\
U_{\delta_5}(0,n_2,0,m_2) &= \tfrac{1}{2}\,\big(1-(-1)^{n_2}+(-1)^{m_2}+(-1)^{n_2+m_2}\big)\ .
\end{aligned} \tag{4.8}$$

For triples of cycles the phases are given by

$$\begin{aligned}
U_{\delta_5}(n_1,0,m_1,m_2) &= \tfrac{1}{2}\,\big((-1)^{m_2}+(-1)^{n_1+m_2}+(-1)^{m_1+m_2}-(-1)^{n_1+m_1+m_2}\big)\ ,\\
U_{\delta_5}(0,n_2,m_1,m_2) &= U_{\delta_5}(0,n_2,0,m_2)\ ,\\
U_{\delta_5}(n_1,n_2,m_1,0) &= U_{\delta_5}(n_1,0,m_1,0)\ ,\\
U_{\delta_5}(n_1,n_2,0,m_2) &= U_{\delta_5}(0,0,n_2,m_2)\ .
\end{aligned} \tag{4.9}$$

The GSO phases for any n_i,m_i, $i=1,2$ are given generally by an expression of the form

$$U_{\delta_5}(\boldsymbol{n},\boldsymbol{m}) \;=\; \alpha\,(-1)^{n_1+n_2+m_1+m_2}+\beta_1\,(-1)^{n_1+n_2+m_1}+\beta_2\,(-1)^{n_1+n_2+m_2}+\beta_3\,(-1)^{n_1+m_1+m_2}$$

$$
\begin{aligned}
&+\beta_4\,(-1)^{n_2+m_1+m_2}+\gamma_1\,(-1)^{n_1+n_2}+\gamma_2\,(-1)^{m_1+m_2}+\gamma_3\,(-1)^{n_1+m_1}\\
&+\gamma_4\,(-1)^{n_2+m_2}+\gamma_5\,(-1)^{n_1+m_2}+\gamma_6\,(-1)^{n_2+m_1}+\varepsilon_1\,(-1)^{n_1}+\varepsilon_2\,(-1)^{n_2}\\
&+\varepsilon_3\,(-1)^{m_1}+\varepsilon_4\,(-1)^{m_2}+\eta\ .
\end{aligned}
\tag{4.10}
$$

The phase (4.10) must reduce to (4.6)–(4.9) when the appropriate winding numbers are set to zero. This gives a system of equations which are enough to determine $U_{\delta_5}(\boldsymbol{n},\boldsymbol{m})$ up to a proportionality constant which may be fixed by requiring the phase to be normalised as ± 1.

One can compute all 16 phase factors in this way as functions of generic thermal winding numbers $\boldsymbol{n},\boldsymbol{m}$. After some inspection and calculation, one finds that as a function $U:\{0,1\}^2\times\mathbb{Z}\to\{\pm 1\}$ the GSO phase (4.5) is given by

$$
\begin{aligned}
U_{\left({\boldsymbol{a}\atop\boldsymbol{b}}\right)}(\boldsymbol{n},\boldsymbol{m}) \;=\;& \tfrac14\,(-1)^{\boldsymbol{a}\cdot\boldsymbol{b}}\Big[(-1)^{n_1+n_2+m_1+m_2}\,(-1)^{a_1+a_2+b_1+b_2}-(-1)^{n_1+n_2+m_1}\,(-1)^{a_1+a_2+b_1}\\
&-(-1)^{n_1+n_2+m_2}\,(-1)^{a_1+a_2+b_2}-(-1)^{n_1+m_1+m_2}\,(-1)^{a_1+b_1+b_2}\\
&-(-1)^{n_2+m_1+m_2}\,(-1)^{a_2+b_1+b_2}+(-1)^{n_1+n_2}\,(-1)^{a_1+a_2}+(-1)^{m_1+m_2}\,(-1)^{b_1+b_2}\\
&-(-1)^{n_1+m_1}\,(-1)^{a_1+b_1}-(-1)^{n_2+m_2}\,(-1)^{a_2+b_2}+(-1)^{n_1+m_2}\,(-1)^{a_1+b_2}\\
&+(-1)^{n_2+m_1}\,(-1)^{a_2+b_1}-(-1)^{n_1}\,(-1)^{a_1}+(-1)^{n_2}\,(-1)^{a_2}\\
&+(-1)^{m_1}\,(-1)^{b_1}+(-1)^{m_2}\,(-1)^{b_2}+1\Big]\ .
\end{aligned}
\tag{4.11}
$$

For our particular calculation the Riemann surface Σ_2 is a branched covering of a torus, and the modified GSO projection is very simple since there is only one homology cycle of the cover which is wrapped around the temperature direction after the reduction to (3.11) given by $(\boldsymbol{n},\boldsymbol{m})\to(0,0,r,0)$. The only even spin structure GSO phases which are non-trivial for generic r are given by

$$
U_{\delta_7}(0,0,r,0)=U_{\delta_8}(0,0,r,0)=U_{\delta_9}(0,0,r,0)=U_{\delta_0}(0,0,r,0)=(-1)^r\ . \tag{4.12}
$$

All other even spin structure phases are equal to 1.

4.3 Chiral Measure

Holomorphic factorization of the genus two superstring measure at zero temperature is achieved by trading the Belavin-Knizhnik obstruction (encoded through the partition function of a free chiral scalar field on Σ_2 given by $(4\pi^2\,\alpha'\,)^{-5}\,(\det\overline{\partial}_0)^{-10}$) for an integral over internal loop momenta $\boldsymbol{p}_\mu\in\mathbb{R}^2$, $\mu=0,1,\dots,9$ flowing through the $\boldsymbol{a}$ cycles of Σ_2 [56]. The resulting dependence on moduli and spin structures is intricately encoded into various sections of the twisted spinor bundles over Σ_2 [57, 24], which may be expressed in terms of modular forms associated with the Riemann surface [19]. The *chiral* measure corresponding to a fixed even spin structure δ is required to be free of global gravitational anomalies on super-moduli space before integrating out the fermionic moduli [18]. On $\mathcal{M}_2$ it may be computed explicitly to be [19]

$$
\mathrm{d}\mu_2[\delta](\Omega)=\left(\frac{1}{4\pi^2\,\alpha'}\right)^2\ \mathrm{d}\Omega_{11}\wedge\mathrm{d}\Omega_{22}\wedge\mathrm{d}\Omega_{12}\ \frac{\Xi_6[\delta](\Omega)\,\Theta[\delta](\mathbf{0}|\Omega)^4}{\Psi_{10}(\Omega)}\ , \tag{4.13}
$$

where Ψ_{10} is the modular form of weight ten defined in (3.35) which arises as the bosonic contribution, and

$$
\Xi_6[\delta](\Omega):=\sum_{1\le k<l\le 3}\langle\nu_{i_k}|\nu_{i_l}\rangle\prod_{j=4,5,6}\Theta[\nu_{i_k}+\nu_{i_l}+\nu_{i_j}](\mathbf{0}|\Omega)^4\ . \tag{4.14}
$$

Here we have chosen a partition $\{i_1,i_2,i_3\}\cup\{i_4,i_5,i_6\}=\{1,2,3,4,5,6\}$ of the index set labelling the odd characteristics (4.2) such that $\delta=\nu_{i_1}+\nu_{i_2}+\nu_{i_3}=\nu_{i_4}+\nu_{i_5}+\nu_{i_6}$. The quantity (4.14) depends only on the spin structure δ and not on the actual triplet of odd characteristics used to represent δ. This follows from the fact that the odd spin structures $\nu_{i_1},\nu_{i_2},\nu_{i_3}$ result from a choice of worldsheet gravitino field insertion points, and the two-loop chiral measure is completely independent of these points [19]. The object $\Xi_6[\delta](\Omega)$ has modular weight 6, but it is *not* a modular form because it depends on the spin structure δ and an additional sign factor arises in its modular transformation laws [19]. As a consequence, the measure (4.13) is modular covariant of weight -5.

The full chiral genus two superstring measure is obtained by summing (4.13) over all even spin structures δ with weights provided by the phases $U_\delta(\boldsymbol{n},\boldsymbol{m})$ computed in Section 4.2 which take into account the modification of the GSO projection at finite temperature. Thus we define

$$\mathrm{d}\mu_2\begin{bmatrix}\boldsymbol{n}\\ \boldsymbol{m}\end{bmatrix}(\Omega)=\sum_{\delta\,\text{even}}U_\delta(\boldsymbol{n},\boldsymbol{m})\ \mathrm{d}\mu_2[\delta](\Omega)\ . \tag{4.15}$$

The quantity $\Upsilon_8(\Omega):=\sum_{\delta\,\text{even}}\Xi_6[\delta](\Omega)\,\Theta[\delta](\boldsymbol{0}|\Omega)^4$ is a uniquely constructed modular form of weight 8. Using the Riemann bilinear relations one can show that [19] $\Upsilon_8(\Omega)=2\,\Psi_8(\Omega)-\frac12\,\Psi_4(\Omega)^2$ where $\Psi_8(\Omega)$ and $\Psi_4(\Omega)$ are respectively the weight 8 and weight 4 generators of the polynomial ring of genus two modular forms. By Igusa's theorem [38], $\Psi_8(\Omega)$ is the unique modular form of weight 8 with $4\,\Psi_8(\Omega)=\Psi_4(\Omega)^2$. It follows that $\Upsilon_8(\Omega)=0$ and thus we have the identity

$$\sum_{\delta\,\text{even}}\Xi_6[\delta](\Omega)\,\Theta[\delta](\boldsymbol{0}|\Omega)^4=0\ . \tag{4.16}$$

Using this along with the modified GSO phases (4.12) corresponding to the reduced form of the period matrix (3.18), we can bring (4.15) to the form

$$\begin{aligned}\mathrm{d}\mu_2\begin{bmatrix}0\\0\\r\\0\end{bmatrix}(\Omega)\ &=\ \left(\frac{1}{4\pi^2\,\alpha'}\right)^2\ \mathrm{d}\Omega_{11}\wedge\mathrm{d}\Omega_{22}\wedge\mathrm{d}\Omega_{12}\ \frac{\left(\mathrm{e}^{\,\pi\,\mathrm{i}\,r}-1\right)}{\Psi_{10}(\Omega)}\\ &\quad\times\Big(\Xi_6[\delta_7](\Omega)\,\Theta[\delta_7](\boldsymbol{0}|\Omega)^4+\Xi_6[\delta_8](\Omega)\,\Theta[\delta_8](\boldsymbol{0}|\Omega)^4\\ &\qquad+\Xi_6[\delta_9](\Omega)\,\Theta[\delta_9](\boldsymbol{0}|\Omega)^4+\Xi_6[\delta_0](\Omega)\,\Theta[\delta_0](\boldsymbol{0}|\Omega)^4\Big)\ .\end{aligned} \tag{4.17}$$

As in the one-loop case, when r is even the Fermi fields are periodic and so the fermions and bosons have the same boundary conditions. These sectors are supersymmetric, and the mode expansions of both the fermion and boson fields contain zero modes. The integration over fermionic zero modes gives zero. Hence (4.17) vanishes, as expected in the supersymmetric sectors.

4.4 Free Energy

The chiral measure (4.15) is a modular form of weight -5. When we include both left and right moving degrees of freedom of the string theory, the non-chiral measure $\mathrm{d}\mu[{\boldsymbol{n}\atop\boldsymbol{m}}](\Omega)\wedge\overline{\mathrm{d}\mu[{\boldsymbol{n}\atop\boldsymbol{m}}](\Omega)}$ is a modular form of weight -10. The complete measure which defines a modular invariant function on moduli space $\mathcal{M}_2$ is thus

$$\mathrm{d}\mu_2\begin{bmatrix}\boldsymbol{n}\\ \boldsymbol{m}\end{bmatrix}(\Omega\,,\,\overline{\Omega})=\left(\det\Omega_2\right)^{-5}\ \mathrm{d}\mu_2\begin{bmatrix}\boldsymbol{n}\\ \boldsymbol{m}\end{bmatrix}(\Omega)\wedge\overline{\mathrm{d}\mu_2\begin{bmatrix}\boldsymbol{n}\\ \boldsymbol{m}\end{bmatrix}(\Omega)}\ . \tag{4.18}$$

We now substitute the full superstring measure (4.18) into (2.6) using (4.17), and resolve the delta-function constraint after performing the necessary reduction to (3.18) (including the appropriate Jacobian). The superstring free energy is thereby found to be

$$
\begin{aligned}
F_2 \;=\;& -\frac{g_s^2}{4}\left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^4 \sum_{N=1}^{\infty}\frac{\mathrm{e}^{-\frac{\beta N}{\sqrt{2}\,R}}}{N}\sum_{\substack{r\,z=N\\ r\,\mathrm{odd}}}\frac{1}{r^4}\sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z}\\ y\neq0}}\int_{\Delta}\frac{\mathrm{d}^2\tau}{(\tau_2)^4}\,\Big|\Psi_{10}(\Omega)\Big|^{-2}\\
&\times\Big|\Xi_6[\delta_7](\Omega)\,\Theta[\delta_7](\mathbf{0}|\Omega)^4+\Xi_6[\delta_8](\Omega)\,\Theta[\delta_8](\mathbf{0}|\Omega)^4\\
&\qquad+\Xi_6[\delta_9](\Omega)\,\Theta[\delta_9](\mathbf{0}|\Omega)^4+\Xi_6[\delta_0](\Omega)\,\Theta[\delta_0](\mathbf{0}|\Omega)^4\Big|^2\ .
\end{aligned}
\tag{4.19}
$$

The quantities (4.14) are worked out in Appendix B. We denote $\Theta_i(\Omega):=\Theta[\delta_i](\mathbf{0}|\Omega)$. Using the explicit expression (3.35), the integrand in (4.19) can be expanded out in terms of the ten even characteristic genus two theta-constants to get

$$
\begin{aligned}
F_2 \;=\;& -\frac{g_s^2}{4}\left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^4 \sum_{N=1}^{\infty}\frac{\mathrm{e}^{-\frac{\beta N}{\sqrt{2}\,R}}}{N}\sum_{\substack{r\,z=N\\ r\,\mathrm{odd}}}\frac{1}{r^4}\\
&\times\sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z}\\ y\neq0}}\int_{\Delta}\frac{\mathrm{d}^2\tau}{(\tau_2)^4}\,\Bigg|4\left(\frac{\Theta_7\,\Theta_8\,\Theta_9\,\Theta_0}{\Theta_1\,\Theta_2\,\Theta_3\,\Theta_4\,\Theta_5\,\Theta_6}\right)^2\\
&+\left(\frac{\Theta_2\,\Theta_3\,\Theta_5\,\Theta_7}{\Theta_1\,\Theta_4\,\Theta_6\,\Theta_8\,\Theta_9\,\Theta_0}\right)^2-\left(\frac{\Theta_1\,\Theta_4\,\Theta_6\,\Theta_7}{\Theta_2\,\Theta_3\,\Theta_5\,\Theta_8\,\Theta_9\,\Theta_0}\right)^2+\left(\frac{\Theta_2\,\Theta_3\,\Theta_6\,\Theta_8}{\Theta_1\,\Theta_4\,\Theta_5\,\Theta_7\,\Theta_9\,\Theta_0}\right)^2\\
&-\left(\frac{\Theta_1\,\Theta_4\,\Theta_5\,\Theta_8}{\Theta_2\,\Theta_3\,\Theta_6\,\Theta_7\,\Theta_9\,\Theta_0}\right)^2+\left(\frac{\Theta_3\,\Theta_4\,\Theta_5\,\Theta_9}{\Theta_1\,\Theta_2\,\Theta_6\,\Theta_7\,\Theta_8\,\Theta_0}\right)^2-\left(\frac{\Theta_1\,\Theta_2\,\Theta_6\,\Theta_9}{\Theta_3\,\Theta_4\,\Theta_5\,\Theta_7\,\Theta_8\,\Theta_0}\right)^2\\
&+\left(\frac{\Theta_3\,\Theta_4\,\Theta_6\,\Theta_0}{\Theta_1\,\Theta_2\,\Theta_5\,\Theta_7\,\Theta_8\,\Theta_9}\right)^2-\left(\frac{\Theta_1\,\Theta_2\,\Theta_5\,\Theta_0}{\Theta_3\,\Theta_4\,\Theta_6\,\Theta_7\,\Theta_8\,\Theta_9}\right)^2\Bigg|^2\ .
\end{aligned}
\tag{4.20}
$$

The theta-constants appearing in (4.20) are functions of the period matrix (3.18) and therefore depend on both the discrete and continuous parameters which characterize the branched covers Σ_2. Their explicit forms in terms of elliptic Jacobi-Erderlyi functions are given by the formula (3.33). We have not found any genus one theta-function identities which could simplify (4.20) and make this expression more explicit.

Note that, in contrast to the one-loop case which relied solely on the Jacobi identity (2.40), the modular invariance of the two-loop thermodynamic free energy does not follow from Riemann identities alone but in addition requires a special property of the ring of modular forms at genus two. The drastic difference between the summation prefactors in the bosonic case (3.36) and in the supersymmetric case (4.20) reflects the different analytic natures of the associated twist field perturbations described in Section 1. This difference will be encountered again in a more explicit form in Section 6.2.

5 Heterotic Strings

Let us now describe how our analysis applies to heterotic string theory. We replace the matter field action $S[X]+\overline{S[X]}$ in (2.1) by

$$S_{\rm het}[X,\lambda]=\frac{1}{4\pi\,\alpha'}\int\limits_{\Sigma_2}{\rm d}^2z\,\Big(|\partial x^\mu|^2+\psi_\mu\,\overline{\partial}\psi^\mu+\lambda_A\,\partial\lambda^A\Big)\ , \tag{5.1}$$

where the fermionic fields λ^A, $A=1,\dots,32$ are Lorentz singlets. Both ψ^μ and λ^A are Majorana-Weyl fermion fields. The ghost contributions are unchanged. Thus the left-moving (holomorphic) part of the heterotic string coincides with that of the superstring whose chiral modular covariant measure is given by (4.17). After bosonization of λ^A, the right-moving (antiholomorphic) part coincides with that of the bosonic string of Section 3.5 with 16 anti-chiral bosons compactified on the Cartan torus of the heterotic gauge group G, where $G=Spin(32)/\mathbb{Z}_2$ or $G=E_8\times E_8$. The compactified bosonic fields produce an extra winding contribution given by a theta-function of the root lattice of G, which at genus two is the unique modular form of weight eight given by [47]

$$\Psi_8(\Omega)=\sum_{\delta\,{\rm even}}\Theta[\delta](\mathbf{0}|\Omega)^{16}\ . \tag{5.2}$$

It follows that the two-loop anti-chiral heterotic string measure is [19]

$${\rm d}\mu_2^{\rm het}\big(\,\overline{\Omega}\,\big)=\left(\frac{1}{4\pi^2\,\alpha'}\right)^6\,{\rm d}\overline{\Omega}_{11}\wedge{\rm d}\overline{\Omega}_{12}\wedge{\rm d}\overline{\Omega}_{22}\ \frac{\overline{\Psi_8(\Omega)}}{\overline{\Psi_{10}(\Omega)}}\ . \tag{5.3}$$

The full modular invariant non-chiral measure is thus $(\det\Omega_2)^{-5}\ {\rm d}\mu_2[{\boldsymbol n\atop\boldsymbol m}](\Omega)\wedge{\rm d}\mu_2^{\rm het}(\,\overline{\Omega}\,)$. Substituting this into (2.6) using (4.17) and (5.3), by proceeding as before we find that the heterotic string free energy is given by

$$\begin{aligned}F_2^{\rm het}\ &=\ \frac{g_s^2}{8}\left(\frac{1}{128\,\sqrt2\,\pi^3\,\alpha'\,\beta\,R}\right)^4\,\sum_{N=1}^\infty\frac{{\rm e}^{-\frac{\beta\,N}{\sqrt2\,R}}}{N}\,\sum_{\substack{r\,z=N\\ r\,{\rm odd}}}\frac{1}{r^4}\,\sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z}\\ y\neq0}}\ \int\limits_\Delta\frac{{\rm d}^2\tau}{(\tau_2)^4}\ \frac{\overline{\Psi_8(\Omega)}}{\big|\Psi_{10}(\Omega)\big|^2}\\ &\quad\times\Big(\Xi_6[\delta_7](\Omega)\,\Theta[\delta_7](\mathbf{0}|\Omega)^4+\Xi_6[\delta_8](\Omega)\,\Theta[\delta_8](\mathbf{0}|\Omega)^4\\ &\qquad+\Xi_6[\delta_9](\Omega)\,\Theta[\delta_9](\mathbf{0}|\Omega)^4+\Xi_6[\delta_0](\Omega)\,\Theta[\delta_0](\mathbf{0}|\Omega)^4\Big)\ .\end{aligned} \tag{5.4}$$

As in (4.20), this expression can be expanded into the ten even characteristic genus two theta-constants by using (3.35), (5.2) and the formulas of Appendix B to get

$$\begin{aligned}F_2^{\rm het}\ &=\ \frac{g_s^2}{8}\left(\frac{1}{128\,\sqrt2\,\pi^3\,\alpha'\,\beta\,R}\right)^4\,\sum_{N=1}^\infty\frac{{\rm e}^{-\frac{\beta\,N}{\sqrt2\,R}}}{N}\,\sum_{\substack{r\,z=N\\ r\,{\rm odd}}}\frac{1}{r^4}\,\sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z}\\ y\neq0}}\ \sum_{i=0}^9\ \int\limits_\Delta\frac{{\rm d}^2\tau}{(\tau_2)^4}\ \frac{\big(\overline{\Theta}_i\big)^{14}}{\prod\limits_{j\neq i}\overline{\Theta}_j} \\ &\quad\times\left[4\left(\frac{\Theta_7\,\Theta_8\,\Theta_9\,\Theta_0}{\Theta_1\,\Theta_2\,\Theta_3\,\Theta_4\,\Theta_5\,\Theta_6}\right)^2+\left(\frac{\Theta_2\,\Theta_3\,\Theta_5\,\Theta_7}{\Theta_1\,\Theta_4\,\Theta_6\,\Theta_8\,\Theta_9\,\Theta_0}\right)^2-\left(\frac{\Theta_1\,\Theta_4\,\Theta_6\,\Theta_7}{\Theta_2\,\Theta_3\,\Theta_5\,\Theta_8\,\Theta_9\,\Theta_0}\right)^2\right.\\ &\quad+\left(\frac{\Theta_2\,\Theta_3\,\Theta_6\,\Theta_8}{\Theta_1\,\Theta_4\,\Theta_5\,\Theta_7\,\Theta_9\,\Theta_0}\right)^2-\left(\frac{\Theta_1\,\Theta_4\,\Theta_5\,\Theta_8}{\Theta_2\,\Theta_3\,\Theta_6\,\Theta_7\,\Theta_9\,\Theta_0}\right)^2+\left(\frac{\Theta_3\,\Theta_4\,\Theta_5\,\Theta_9}{\Theta_1\,\Theta_2\,\Theta_6\,\Theta_7\,\Theta_8\,\Theta_0}\right)^2\\ &\quad\left.-\left(\frac{\Theta_1\,\Theta_2\,\Theta_6\,\Theta_9}{\Theta_3\,\Theta_4\,\Theta_5\,\Theta_7\,\Theta_8\,\Theta_0}\right)^2+\left(\frac{\Theta_3\,\Theta_4\,\Theta_6\,\Theta_0}{\Theta_1\,\Theta_2\,\Theta_5\,\Theta_7\,\Theta_8\,\Theta_9}\right)^2-\left(\frac{\Theta_1\,\Theta_2\,\Theta_5\,\Theta_0}{\Theta_3\,\Theta_4\,\Theta_6\,\Theta_7\,\Theta_8\,\Theta_9}\right)^2\right]\ ,\end{aligned} \tag{5.5}$$

which can again be expressed in terms of elliptic Jacobi-Erderlyi functions by using the formula (3.33).

In the free string limit $g_s \to 0$ the space of physical states of the heterotic sigma-model on the symmetric product orbifold (1.7) is naturally isomorphic to the Fock space of second quantized heterotic strings in DLCQ [52, 43]. The $(\mathbb{Z}_2)^N$ factor in this quotient space is a discrete gauge symmetry acting on twisted sector gauge fermions λ^A in the fundamental representation of G. The additional $\mathbb{Z}_2$-orbifolds are achieved by extra GSO projections on λ^A, and they are necessary to reproduce the light-cone Green-Schwarz heterotic string field theory [52, 43]. The right-moving sector is thus given by the standard $\mathbb{R}^{24}/\mathbb{Z}_2$ orbifold conformal field theory. This $\mathbb{Z}_2$-orbifold for $g_s > 0$ is manifested through the decomposition of the theta-constants comprising the modular form (5.2) according to (3.33), and it can be thought of as being ultimately responsible in this instance for the fibred decomposition of the Jacobian variety (3.19). Let us also remark that in order to implement S-duality with Type IB superstring theory (as is necessary in formulating the heterotic matrix string theory conjecture), one should add a Wilson line which breaks the heterotic gauge group G to $SO(16) \times SO(16)$ [52, 43]. This may be achieved by adding an appropriate B-field term $\lambda^A \, B_{AB} \, \lambda^B$ to the heterotic string action (5.1), whose effect is to simply modify the modular form (5.2) in a standard way. It amounts to a shift of the imaginary part Ω_2 of the period matrix of Σ_2 and thus produces a reduction onto different tori in the right-moving sector. This (non-modular) change of the base tori can be derived directly from the corresponding Polyakov path integral [32].

6 Boundary Contributions

In this final section we will elucidate some arithmetic and physical aspects of the two-loop superstring free energy (4.20). We have seen that the pertinent genus two theta-functions (3.33) factorize into elliptic Jacobi-Erderlyi functions associated with the fibration (3.19) of the Jacobian variety of the original curve Σ_2 into two tori $\mathbb{T}^2_{\mathrm{i}\nu}$ and $\mathbb{T}^2_{\tau}$. But the resulting formulas for the free energies are quite involved and difficult to deal with analytically. We will now explore some regions of the moduli space $\mathcal{M}_2$ wherein this factorization simplifies drastically and some precise information can be extracted from these expressions.

6.1 Pinching Parameters

Let us begin with some general aspects concerning the generic relationship between genus two curves and elliptic curves. Generally, any genus two surface Σ_2 is a connected sum $\Sigma_2 = \mathbb{T}^2_{\tau_1} \# \mathbb{T}^2_{\tau_2}$ of two tori whose periods can be expressed in terms of the moduli τ_i of the tori and a complex number t. The positive number $|t| < 1$ is the radius of the disks that are excised from the two tori in order to sew them together to produce Σ_2. Let $q_i := \mathrm{e}^{\,2\pi\,\mathrm{i}\,\tau_i}$, $i = 1, 2$. Then the pinching parameters q_1, q_2, t form an alternative set of moduli for Σ_2.

The genus two period matrix Ω may be computed as a holomorphic function of the pinching parameters q_1, q_2, t [59]. For this, we use the sewing formalism to express the holomorphic one-differential $\boldsymbol{\omega}$ of Σ_2 as a power series in t with coefficients calculated from the genus one differentials $\omega^{(i)}$ of $\mathbb{T}^2_{\tau_i}$, $i = 1, 2$, and then use (2.4) to calculate the period matrix elements. We will need these expressions only to leading order in $t \to 0$, in which case the period matrix is

given by

$$
\begin{aligned}
\Omega_{11} &= \tau_1 + \frac{t^2}{2\pi\,\mathrm{i}}\,\hat{E}_2(q_1) + O\left(t^4\right) \ , \\
\Omega_{22} &= \tau_2 + \frac{t^2}{2\pi\,\mathrm{i}}\,\hat{E}_2(q_2) + O\left(t^4\right) \ , \\
\Omega_{12} &= -\frac{t}{2\pi\,\mathrm{i}}\left(1 + \hat{E}_2(q_1)\,\hat{E}_2(q_2)\,t^2\right) + O\left(t^5\right)
\end{aligned} \tag{6.1}
$$

where

$$
\hat{E}_2(q) = -\frac{1}{12} + 2\sum_{n=1}^{\infty} \sigma_1(n)\ q^n \tag{6.2}
$$

is the normalized elliptic Eisenstein series, with $\sigma_1(n)$ the number of n-sheeted unbranched covers of a torus given by (2.27). Let us now specialize to the case where $\Sigma_2 \to \mathbb{T}^2_{\mathrm{i}\,\nu}$ is a branched covering with the reduced form (3.18) of its period matrix. The moduli of the two connecting tori can then be identified as $\tau_1 = (x + \mathrm{i}\,\frac{r}{\nu})/z$ and $\tau_2 = -\Omega_{22} = \tau$. The torus $\mathbb{T}^2_{\tau_1}$ in this case is an unbranched cover of the base $\mathbb{T}^2_{\mathrm{i}\,\nu}$ of degree $N = r\,z$. The radius of the connecting cylinder may be identified as $|t| = \frac{y}{z}$, which satisfies $0 < |t| < 1$ since $y \in \mathbb{Z}/z\,\mathbb{Z}$ and $y \neq 0$.

There are two classes of degenerations of the Riemann surface Σ_2 up to modular transformations. When $t \to 0$, the connecting cylinder is pinched down and Σ_2 degenerates into the two tori $\mathbb{T}^2_{\tau_1}$ and $\mathbb{T}^2_{\tau_2}$. This provides a geometric description of the moduli space $\mathcal{M}_2$ near the divisor of surfaces Σ_2 with nodes, and it corresponds to the limit in which the two branch points on $\mathbb{T}^2_{\mathrm{i}\,\nu}$ coincide (singularity type (b) in the terminology of Section 3.1). When $q_i \to 0$ for $i = 1$ or $i = 2$, i.e. $\tau_i \to \mathrm{i}\,\infty$, the torus $\mathbb{T}^2_{\tau_i}$ degenerates to a Riemann sphere by making its homology cycle β infinitely long, or equivalently by modular invariance shrinking the cycle to zero size (singularity type (c)). It is straightforward to see that the other boundary limits of the moduli space $\mathcal{M}_2$, determined by the positivity condition (3.24), can be mapped into these other two cases. Let us now examine each of these limits in some detail.

6.2 Factorization

In the sewing construction one may view the genus two surface Σ_2 as the disjoint union $\Sigma_2 = \mathbb{E}_1 \amalg \mathbb{A}_t \amalg \mathbb{E}_2$, where $\mathbb{A}_t = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_i \in \mathbb{B}^2\ ,\ z_1\,z_2 = t\}$ for $t \neq 0$ is the annulus with outer radius 1 and inner radius t, $\mathbb{B}^2$ is the unit disk in $\mathbb{C}$, and $\mathbb{E}_i = \mathbb{T}^2_{\tau_i} \setminus \mathbb{B}^2$ with z_i local complex coordinates on $\mathbb{T}^2_{\tau_i}$. In conformal field theory, the surfaces with boundary $\mathbb{E}_i$, $i = 1, 2$ define two states $\langle \mathbb{E}_1|$ and $|\mathbb{E}_2\rangle$. Within the Hamiltonian framework, we identify the annulus with a cylinder via the exponential map. The cylinder amplitude then corresponds to the operator insertion $t^{L_0}\,\bar{t}^{\,\overline{L}_0}$, where $L_0 + \overline{L}_0$ is the worldsheet Hamiltonian and $L_0 - \overline{L}_0$ is the momentum operator.

The genus two superstring free energy is then given symbolically by

$$
F_2 = \left\langle \mathbb{E}_1 \right| t^{L_0}\,\bar{t}^{\,\overline{L}_0} \left| \mathbb{E}_2 \right\rangle \ . \tag{6.3}
$$

We can insert a complete set of states into the matrix element (6.3) which diagonalize the Virasoro operators $L_0, \overline{L}_0$ to get

$$
F_2 = \sum_I \left\langle \mathbb{E}_1 \middle| \psi_I \right\rangle \left\langle \psi_I \right| t^{L_0}\,\bar{t}^{\,\overline{L}_0} \left| \psi_I \right\rangle \left\langle \psi_I \middle| \mathbb{E}_2 \right\rangle \ . \tag{6.4}
$$

This yields a Laurent series expansion in $|t|$. After GSO projection, the leading contribution comes from the massless vacuum states having $L_0 = \overline{L}_0 = 0$ and zero momentum, so that

$$F_2 = F^{(1)}\, F^{(2)} + O\big(|t|\big) \ , \tag{6.5}$$

where $F^{(i)}$ is the one-loop free energy for the torus $\mathbb{T}^2_{\tau_i}$. We should stress that the expression (6.5) is only meant to be symbolic. In particular, it is only valid at fixed spin structure and fixed winding numbers around the finite temperature DLCQ torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$, in which case the leading term is actually down by a negative power of $|t|$. Summing over these quantum numbers mixes the two one-loop contributions in a non-trivial way and spoils the explicit factorization of the leading order term. We shall see this explicitly below. The higher-order terms in (6.5) arise from propagation of massless physical states in the long thin tube connecting the two tori [6], and in this limit the genus two free energy is related to a sum of products of one-loop tadpoles for the massless states represented as torus one-point functions.

We will now identify these one-loop string theories. Let $\delta = \binom{\boldsymbol{a}_1}{\boldsymbol{a}_2} \neq \delta_0$ be any even genus two spin structure such that $\boldsymbol{a}_i \in \{0,1\}^2$ is an even genus one spin structure on $\mathbb{T}^2_{\tau_i}$. In the limit $t \to 0$, the leading asymptotics of the genus two theta-constants are given by

$$\begin{aligned} \Theta[\delta](\mathbf{0}|\Omega) &= \theta[\boldsymbol{a}_1](0|\tau_1)\,\theta[\boldsymbol{a}_2](0|\tau_2) + O\left(t^2\right) \ , \\ \Theta[\delta_0](\mathbf{0}|\Omega) &= t\ \eta(\tau_1)^3\,\eta(\tau_2)^3 + O\left(t^3\right) \ , \end{aligned} \tag{6.6}$$

which implies that the cusp form (3.35) has the leading asymptotics

$$\Psi_{10}(\Omega) = t^2\ \eta(\tau_1)^{24}\,\eta(\tau_2)^{24} + O\left(t^4\right) \ . \tag{6.7}$$

It is instructive to first examine the behaviour of the bosonic free energy (3.36) in this limit. Notice, first of all, that since $y \in \mathbb{Z}/z\,\mathbb{Z}$ with $y \neq 0$, the limit $t \to 0$ is equivalent to taking $z \to \infty$, i.e. the limit $N \to \infty$ of branched covers with large degree. This means that we should look at the large N asymptotic tail behaviour of the series (3.36). One then has

$$\begin{aligned} \lim_{z\to\infty}\ F_2^{\rm bos} &= -g_s^2\left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^{12}\ \sum_{N=1}^{\infty}\ \mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}\ \sum_{r\,z=N}\ \left(\frac{z}{r}\right)^{12} \\ &\quad\times \sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z}\\ y\neq 0}}\ \frac{1}{y^4}\ \mathcal{Z}_1^{\rm bos}\big(\tau'\,,\,\overline{\tau}\,'\big)\Big|_{\tau'=\frac{x+\frac{\mathrm{i}\,r}{\nu}}{z}}\ \tilde{F}_1^{\rm bos} \end{aligned} \tag{6.8}$$

where

$$\tilde{F}_1^{\rm bos} = \int\limits_{\Delta} \frac{\mathrm{d}^2\tau}{(\tau_2)^{12}}\ \mathcal{Z}_1^{\rm bos}(\tau,\overline{\tau}\,) \tag{6.9}$$

and $\mathcal{Z}_1^{\rm bos}(\tau,\overline{\tau}\,) =\ \mathrm{Tr}\ q^{L_0-2}\,\overline{q}^{\,\overline{L}_0-2} = |\eta(\tau\,)|^{-48}$ is the one-loop first quantized partition function on $\mathbb{T}^2_\tau$. Thus the contribution of the unramified coverings of $\mathbb{T}^2_{\mathrm{i}\,\nu}$ is the same as in the one-loop computation of Section 2.3, while the contribution over the auxilliary torus $\mathbb{T}^2_\tau$ resembles the second quantized one-loop bosonic partition function (Note that this is *not* the standard $SL(2,\mathbb{Z})$ modular invariant partition function, as modular invariance of the expression (6.8) under the genus two residual modular group $\mathcal{G} \subset Sp(4,\mathbb{Z})$ is required here). This is a twisted admixture of the operation providing the mapping from first quantization to second quantization that was given by Hecke transforms in Section 2.3.

To understand the algebraic meaning of the mapping in the present case, we now turn our attention to the superstring free energy (4.20). For this, we also need the asymptotic behaviours of the quantities (4.14), which from (6.6) and the formulas of Appendix B can be computed to be

$$
\begin{aligned}
\Xi_6[\delta](\Omega) &= -2^8 \left\langle \boldsymbol{a}_1 \middle| \tbinom{1}{1} \right\rangle \left\langle \boldsymbol{a}_2 \middle| \tbinom{1}{1} \right\rangle \eta(\tau_1)^{12}\, \eta(\tau_2)^{12} + O\left(t^2\right) \ , \\
\Xi_6[\delta_0](\Omega) &= -3 \cdot 2^8 \ \eta(\tau_1)^{12}\, \eta(\tau_2)^{12} + O\left(t^2\right) \ .
\end{aligned} \tag{6.10}
$$

Substituting (6.6) and (6.10) into the numerator of the integrand in (4.19), one finds that the contributions from the spin structures δ_7, δ_8 and δ_9 sum to 0 by the Jacobi abstruse identity (2.40). This sum is tantamount to a partial GSO projection which removes the would be tachyonic divergence coming from (6.7) in the degeneration limit $t \to 0$. Only the contribution from the spin structure δ_0 remains, and (4.19) becomes

$$
\begin{aligned}
\lim_{z\to\infty} F_2 &= -\frac{g_s^2}{4} \left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^4 \sum_{N=1}^\infty \frac{\mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}}{N} \sum_{\substack{r\,z=N \\ r\,\mathrm{odd}}} \frac{1}{r^4} \sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z} \\ y\neq 0}} \int_\Delta \frac{\mathrm{d}^2\tau}{(\tau_2)^4} \left(\frac{3\pi^2}{4}\, \frac{y^2}{z^2}\right)^2 \\
&= -\frac{\sqrt{3}\,\pi^2\, g_s^2}{8} \left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^4 \sum_{N=1}^\infty \frac{\mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}}{N} \sum_{\substack{r\,z=N \\ r\,\mathrm{odd}}} \frac{1}{r^4} \left(\tfrac{1}{5}\, z^2 - \tfrac{1}{2}\, z\right)
\end{aligned} \tag{6.11}
$$

to $O(z^{-1})$. The removal of the tachyonic divergence from $\tilde{F}_2^{\mathrm{bos}}$ in (6.8) has completely trivialized the partition function over the auxilliary torus and the only contribution that remains is from the unbranched cover over $\mathbb{T}^2_{\mathrm{i}\,\nu}$. The precise form of this sum is now determined by the way in which we analyse the large degree asymptotics as $N \to \infty$ of this series.

Let us first take the limit $z \to \infty$ with r finite. In this limit $\tau_1 \to 0$ and the covering torus Σ_1 shrinks to a point. Nevertheless, some remnant of the genus two covering map remains due to the fibration over the auxilliary torus $\mathbb{T}^2_\tau$ in (3.19). In this regime we may disregard the odd parity constraint on the sum over the divisors r in (6.11), and the free energy thereby becomes

$$
\lim_{\substack{z\to\infty \\ r\ll z}} F_2 = -\frac{\sqrt{3}\,\pi^2\, g_s^2}{8} \left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^4 \sum_{N=1}^\infty \frac{\mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}}{N^5} \left(\tfrac{1}{5}\,\sigma_6(N) - \tfrac{1}{2}\,\sigma_5(N)\right) \tag{6.12}
$$

where the divisor functions

$$
\sigma_k(N) = \sum_{z|N} z^k \tag{6.13}
$$

generalize the integers $\sigma_1(N)$ in (2.27) which count the unramified coverings Σ_1 of $\mathbb{T}^2_{\mathrm{i}\,\nu}$. The series (6.12) can be naturally related to the Hecke algebra as follows.

Consider the lattice $\Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\,\tau$ such that $\mathbb{T}^2_\tau = \mathbb{C}/\Lambda_\tau$. For any integer $k \geq 2$, introduce the holomorphic Eisenstein series [3]

$$
G_{2k}(\tau) := \sum_{\substack{\lambda\in\Lambda_\tau \\ \lambda\neq(0,0)}} \frac{1}{\lambda^{2k}} = 2\,\zeta(2k) + 2\,\frac{(2\pi\,\mathrm{i}\,)^k}{(2k-1)!} \sum_{n=1}^\infty \sigma_{2k-1}(n)\ q^n \tag{6.14}
$$

with $q = \mathrm{e}^{\,2\pi\,\mathrm{i}\,\tau}$. This defines a modular form of weight $2k$. The action on (6.14) of the Hecke operator $\mathbf{H}_N$ defined in (2.43) is given by

$$
\mathbf{H}_N * G_{2k}(\tau) = N^{2k-1} \sum_{\substack{\Lambda'\subset\Lambda_\tau \\ [\Lambda_\tau:\Lambda']=N}} \ \sum_{\substack{\lambda\in\Lambda' \\ \lambda\neq(0,0)}} \frac{1}{\lambda^{2k}} \ . \tag{6.15}
$$

To work out this sum explicitly, suppose first that $N=p$ is a prime number. If $\lambda\in p\,\Lambda_\tau$, then λ lies in all sublattices Λ' of Λ_τ of index p and so contributes $\frac{\sigma_1(p)}{\lambda^{2k}}=\frac{p+1}{\lambda^{2k}}$ to the sum (6.15). Otherwise, λ lies in only one sublattice $\Lambda'=p\,\Lambda_\tau\oplus\mathbb{Z}\,\lambda$ and so contributes $\frac{1}{\lambda^{2k}}$. Thus

$$\mathbf{H}_p*G_{2k}(\tau)=p^{2k-1}\,G_{2k}(\tau)+p^{2k}\sum_{\substack{\lambda\in p\,\Lambda_\tau\\ \lambda\neq(0,0)}}\frac{1}{\lambda^{2k}}=p^{2k-1}\,G_{2k}(\tau)+G_{2k}(\tau) \tag{6.16}$$

and it follows that $G_{2k}(\tau)$ is an eigenform of $\mathbf{H}_p$ with eigenvalue $\sigma_{2k-1}(p)=1+p^{2k-1}$. In the general case, we use the prime factorization of the integer N along with the Hecke algebra property $\mathbf{H}_n\circ\mathbf{H}_m=\mathbf{H}_{n\,m}$ for $\gcd(n,m)=1$ to conclude that the Eisenstein series G_{2k} is a simultaneous eigenform of each Hecke operator $\mathbf{H}_N$ with eigenvalue $\sigma_{2k-1}(N)$. Similarly, each $\mathbf{H}_N$ has eigenforms comprised of elliptic cusp forms $\eta(\tau)^{24}\,G_{2k}(\tau)$ [3].

Let us now take the limit $z\to\infty$ with $r\sim z$. In this limit $\tau_1\to\frac{\mathrm{i}}{\nu}$ and the surface Σ_2 factorizes into the original spacetime torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$ (up to a modular transformation) and the auxilliary torus $\mathbb{T}^2_\tau$. The free energy (6.11) in this regime vanishes,

$$\lim_{\substack{z\to\infty\\ r\sim z}}F_2=0\ , \tag{6.17}$$

to leading order. At this order supersymmetry is restored by the factorization and there are no contributions from this boundary component of the moduli space $\mathcal{M}_2$. The combinatorics of the covers in these factorizing degeneration limits are thereby accounted for by a sort of “topological” string theory which counts particular eigenvalues in the spectra of the Hecke operators. The role of the degenerate free energy as a generating function for the Hecke spectra will also persist at higher orders in the cylindrical length t. For example, the Siegel cusp form of weight ten has has the leading expansion [55]

$$\Psi_{10}(\Omega)^{-1}=t^{-2}\,\eta(\tau_1)^{-24}\,\eta(\tau_2)^{-24}\,\Big[1+12\,t^2\,\hat{E}(q_1)\,\hat{E}(q_2)+O\left(t^4\right)\Big] \tag{6.18}$$

as $t\to0$.

6.3 Collapsing Homology Cycles

Let us now look at the limit $q_1\to0$ in which the handle with homology cycles a_1,b_1 degenerates. In this non-separating degeneration limit, the surface Σ_2 becomes the auxilliary torus $\mathbb{T}^2_\tau$. If $\boldsymbol{a}\in\{0,1\}^2$ is any even genus one characteristic, then the even characteristic genus two theta-constants generally have a power series expansion around $q_1=0$ given by

$$\begin{aligned}
\Theta\big[{}^{\boldsymbol{a}}_{00}\big](\mathbf{0}|\Omega) &= \sum_{n=-\infty}^{\infty}(q_1)^{n^2}\ \theta[\boldsymbol{a}]\big(-\tfrac{n\,t}{2\pi\,\mathrm{i}}\big|\tau_2\big)\ ,\\
\Theta\big[{}^{\boldsymbol{a}}_{01}\big](\mathbf{0}|\Omega) &= \sum_{n=-\infty}^{\infty}(-1)^n\,(q_1)^{n^2}\ \theta[\boldsymbol{a}]\big(-\tfrac{n\,t}{2\pi\,\mathrm{i}}\big|\tau_2\big)\ ,\\
\Theta\big[{}^{\boldsymbol{a}}_{10}\big](\mathbf{0}|\Omega) &= \sum_{n=-\infty}^{\infty}(q_1)^{(n+\frac12)^2}\ \theta[\boldsymbol{a}]\big(-\tfrac{(n+\frac12)\,t}{2\pi\,\mathrm{i}}\big|\tau_2\big)\ ,\\
\Theta\big[\delta_0\big](\mathbf{0}|\Omega) &= \sum_{n=-\infty}^{\infty}\mathrm{i}\,(-1)^n\,(q_1)^{(n+\frac12)^2}\ \theta_1\big(-\tfrac{(n+\frac12)\,t}{2\pi\,\mathrm{i}}\big|\tau_2\big)\ .
\end{aligned} \tag{6.19}$$

It follows that the cusp form (3.35) has the leading asymptotics

$$\Psi_{10}(\Omega) = -(q_1)^2\,\eta(\tau_2)^{18}\,\theta_1\big(-\tfrac{t}{4\pi\,\mathrm{i}}\big|\tau_2\big)^2 + O\left((q_1)^2\right)\ . \tag{6.20}$$

After some algebra using the formulas of Appendix B, one thereby finds that leading behaviour of the free energy (4.19) is given by

$$\begin{aligned}\lim_{q_1\to 0} F_2 \ =\ & -\frac{g_s^2}{32}\left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^4 \sum_{N=1}^{\infty}\mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}} \sum_{\substack{r\,z=N\\ r\,\mathrm{odd}}}\frac{1}{r^5}\sum_{\substack{y\in\mathbb{Z}/z\,\mathbb{Z}\\ y\neq 0}}\int_{\Delta}\frac{\mathrm{d}^2\tau}{(\tau_2)^4}\,\frac{1}{\big|\eta(\tau)\big|^{36}\,\big|\theta_1\big(\frac{y}{2\,z}\big|\tau\big)\big|^4}\\ &\times\Big|\theta_4(0|\tau)^8\,\Big[\theta_4\big(\tfrac{y}{2\,z}\big|\tau\big)^4\,\theta_1\big(\tfrac{y}{2\,z}\big|\tau\big)^4+\theta_2\big(\tfrac{y}{2\,z}\big|\tau\big)^4\,\theta_3\big(\tfrac{y}{2\,z}\big|\tau\big)^4\Big]\\ &+\theta_2(0|\tau)^8\,\Big[\theta_4\big(\tfrac{y}{2\,z}\big|\tau\big)^4\,\theta_3\big(\tfrac{y}{2\,z}\big|\tau\big)^4+\theta_2\big(\tfrac{y}{2\,z}\big|\tau\big)^4\,\theta_1\big(\tfrac{y}{2\,z}\big|\tau\big)^4\Big]\\ &-\ \theta_3(0|\tau)^8\,\Big[\theta_1\big(\tfrac{y}{2\,z}\big|\tau\big)^4\,\theta_3\big(\tfrac{y}{2\,z}\big|\tau\big)^4+\theta_2\big(\tfrac{y}{2\,z}\big|\tau\big)^4\,\theta_4\big(\tfrac{y}{2\,z}\big|\tau\big)^4\Big]\Big|^2\ . \end{aligned}\tag{6.21}$$

The elliptic modular integrals in (6.21) are finite.

The degeneration limit $q_1 \to 0$ corresponds to the shrinking limit $\nu \to 0$ of the original spacetime torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$. There are two ways in which we can make the parameter (2.7) vanish. Taking $\beta \to \infty$ gives the zero temperature limit of the free energy, which is proportional to the vacuum energy. Since $N \geq 1$, all terms in the series are exponentially damped and thus the vacuum energy vanishes, as expected since this limit simply corresponds to the restoration of supersymmetry at zero temperature. On the other hand, taking $R \to \infty$ decompactifies the light cone and sends the exponential factors to 1 in (6.21). Apart from an overall factor, the free energy is then independent of temperature, except for its dependence on the winding number r. In this case the strings effectively propagate on a $\mathbb{Z}_2$ orbifold of flat space [2] defined by the antiperiodic fermion boundary conditions, which is presumably a subsector of the symmetric orbifold superconformal field theory on $\mathbb{R}^8$ for each N. This string theory is non-supersymmetric and hence has a non-vanishing vacuum energy corresponding to contributions from physical tachyons [6]. In each of these decompactification limits, the discrete data of the branched cover should assemble themselves into a continuum limit which restores the two complex dimensions of the moduli space $\mathcal{M}_2$ [14].

Let us now consider the non-separating degeneration limit $q_2 \to 0$ in which the branched cover Σ_2 becomes an unramified covering of the original spacetime torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$ (up to a modular transformation). This corresponds to the contributions from the $\tau \to \mathrm{i}\,\infty$ region of the elliptic modular integral in (4.19). We may use the same asymptotic formulas (6.19) and (6.20) with q_1, τ_2 replaced by q_2, τ_1. The terms $\Xi_6[\delta_i]\,\Theta[\delta_i](\mathbf{0}|\Omega)^4$ for $i = 7, 8$ have leading terms of order q_2. These two terms thus give a contribution to the integration over moduli space which has a simple pole at $q_2 = 0$. This divergence arises from the tachyon traversing the a_2 cycle of the elliptic component $\mathbb{T}^2_\tau$ of the degeneration [6, 19]. However, the sum $\Xi_6[\delta_7]\,\Theta[\delta_7](\mathbf{0}|\Omega)^4 + \Xi_6[\delta_8]\,\Theta[\delta_8](\mathbf{0}|\Omega)^4$ is found to vanish to this order and thus removes the pole. This corresponds to a partial GSO projection in the Neveu-Schwarz sector of the genus one component $\mathbb{T}^2_\tau$ which eliminates the tachyon. The contributions from the remaining spin structures δ_0 and δ_9 correspond to Ramond states propagating in $\mathbb{T}^2_\tau$ and are of order $(q_2)^2$, yielding no poles.

Working out each of the four contributions to (4.19) up to order $(q_2)^2$ leads after some algebra

to the free energy

$$\lim_{q_2\to 0} F_2 = -\frac{g_s^2}{64}\left(\frac{1}{4\sqrt{2}\,\pi\,\beta\,R}\right)^4 \sum_{N=1}^\infty \frac{\mathrm{e}^{-\frac{\beta\,N}{\sqrt{2}\,R}}}{N} \sum_{\substack{r\,z=N\\ r\,\mathrm{odd}}} \frac{1}{r^4} \sum_{\substack{x,y\in\mathbb{Z}/z\,\mathbb{Z}\\ y\neq 0}} \left|\mathcal{Z}_1^\infty(\zeta|\tau_1)\right|^2\Bigg|_{\substack{\zeta=\frac{y}{z}\\ \tau_1=\frac{x+\mathrm{i}\,\frac{r}{\nu}}{z}}} \ , \tag{6.22}$$

where

$$\begin{aligned}
\mathcal{Z}_1^\infty(\zeta|\tau_1) \ = \ & \frac{1}{\eta(\tau_1)^{18}\,\theta_1\big(\frac{\zeta}{2}\big|\tau_1\big)^2}\Big[2\,\theta_2(0|\tau_1)^8\,\Big(\theta_1\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\theta_2\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4+\theta_3\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\theta_4\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\Big)\\
& -\theta_3(0|\tau_1)^8\,\Big(\theta_1\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\theta_3\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4+\theta_2\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\theta_4\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\Big)\\
& +\theta_4(0|\tau_1)^8\,\Big(\theta_1\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\theta_4\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4+\theta_2\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\theta_3\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\Big)\\
& -8\,\eta(\tau_1)^3\,\theta_1\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\Big(\theta_2(0|\tau_1)\,\theta_3(0|\tau_1)\,\theta_4(\zeta|\tau_1)+\theta_2(0|\tau_1)\,\theta_4(0|\tau_1)\,\theta_3(\zeta|\tau_1)\\
& \qquad\qquad +\theta_3(0|\tau_1)\,\theta_4(0|\tau_1)\,\theta_2(\zeta|\tau_1)\Big)\\
& -8\,\eta(\tau_1)^3\,\theta_2\big(\tfrac{\zeta}{2}\big|\tau_1\big)^4\,\Big(\theta_2(0|\tau_1)\,\theta_3(0|\tau_1)\,\theta_4(\zeta|\tau_1)+\theta_2(0|\tau_1)\,\theta_4(0|\tau_1)\,\theta_3(\zeta|\tau_1)\\
& \qquad\qquad -\theta_3(0|\tau_1)\,\theta_4(0|\tau_1)\,\theta_2(\zeta|\tau_1)\Big)\Big]
\end{aligned} \tag{6.23}$$

and we have dropped an irrelevant overall numerical constant in (6.22) arising from the remaining modular integration over $\tau_2\in\Delta$. As before, the non-vanishing of this boundary contribution is due to the presence of physical tachyons. This free energy is a natural extension of the one-loop result of Section 2.3, illustrating the appropriate modification for the action of the Hecke algebra at two-loops.

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Appendix A   Moduli Space for the Poincaré Normal Form

In this appendix we will sketch the computation of the two-loop free energy from the fully reduced Poincaré normal form (2.22). This is done for the sake of completeness and because it provides some interesting alternative characterizations of the genus two Hurwitz moduli space which may be of independent interest. As we will see, the free energy in this case cannot be made as explicit as in the main text, but the same reduction features do carry through nonetheless.

A.1 Reduced Moduli

The genus two Poincaré normal form is given by

$$\mathsf{P} = r \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s & t & 0 \end{pmatrix} . \tag{A.1}$$

The matrix T which appears in the Frobenius normal form (2.21) can be absorbed into the period matrix as in (3.14) but the symplectic unimodular matrix

$$\mathsf{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{A.2}$$

which acts on the base torus $\mathbb{T}^2_{\mathrm{i}\,\nu}$ as a modular transformation, remains [45, 39]. This is one of the reasons why the full reduction is undesirable, as both the moduli space and the GSO projection depend explicitly on the four integers a, b, c, d which are functions of the parameters r, s and t. We have to keep S explicitly in all of our calculations, and then sum over all the corresponding $SL(2, \mathbb{Z})$ modular transformations of the base. On the other hand, the full reduction to (A.1) leads to a somewhat simpler decomposition of genus two theta-constants into elliptic theta-functions [11].

Given the homology matrix (2.21), we can rewrite the constraint equation (3.2) using (A.1) and (A.2) to get

$$\mathsf{H}^\top (\mathbb{1}_2, \Omega) = (1, \mathrm{i}\,\nu)\, \mathsf{S}\,\mathsf{P}\,\mathsf{T} = (1, \mathrm{i}\,\nu) \begin{pmatrix} r\,a & r\,b\,s & r\,b\,t & 0 \\ r\,c & r\,d\,s & r\,d\,t & 0 \end{pmatrix} \mathsf{T} . \tag{A.3}$$

In order to factorize the genus two theta-constants in terms of elliptic functions as in Section 3.4, the period matrix must have rational-valued off-diagonal elements. This will happen if we modify (A.3) by multiplying P with the intersection form $-\mathsf{J}_2 = (\mathsf{J}_2)^{-1}$ to obtain

$$\mathsf{H}^\top (\mathbb{1}_2, \Omega) = (1, \mathrm{i}\,\nu) \begin{pmatrix} r\,b\,t & 0 & -r\,a & -r\,b\,s \\ r\,d\,t & 0 & -r\,c & -r\,d\,s \end{pmatrix} \mathsf{J}_2\,\mathsf{T} . \tag{A.4}$$

The matrix $\mathsf{J}_2\,\mathsf{T} \in Sp(4, \mathbb{Z})$ is invertible. The inverse $(\mathsf{J}_2\,\mathsf{T})^{-1}$ acts on the left-hand side of (A.4) as a modular transformation on the period matrix Ω and on the pullback matrix H as in (3.14).

We can now solve the constraint equation for the period matrix by first computing

$$\mathsf{H} = (1, \mathrm{i}\,\nu) \begin{pmatrix} r\,b\,t & 0 \\ r\,d\,t & 0 \end{pmatrix} = r\,t\,(b + d\,\mathrm{i}\,\nu, 0) \tag{A.5}$$

to get

$$\mathsf{H}\,\Omega = r\,t\,(b + d\,\mathrm{i}\,\nu, 0) \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{pmatrix} = (1, \mathrm{i}\,\nu) \begin{pmatrix} -r\,a & -r\,b\,s \\ -r\,c & -r\,d\,s \end{pmatrix} . \tag{A.6}$$

After a $\mathbb{Z}_2$ reflection, we thereby find

$$\Omega = \begin{pmatrix} \frac{\tau_\nu}{\mu\,s} & \frac{1}{\mu} \\ \frac{1}{\mu} & \tau \end{pmatrix} \tag{A.7}$$

where the integer μ is defined through $t = \mu\, s$ and is related to the degree N of the cover by $N = r^2\, \mu\, s$. As before $\tau := \Omega_{22} \in \mathcal{H}_1$ parametrizes an auxilliary torus $\mathbb{T}^2_\tau$, while

$$\tau_\nu = \frac{a + c\,(\mathrm{i}\,\nu)}{b + d\,(\mathrm{i}\,\nu)} \tag{A.8}$$

labels a torus $\mathbb{T}^2_{\tau_\nu}$ in the same elliptic modular orbit as the base $\mathbb{T}^2_{\mathrm{i}\,\nu}$. This is in contrast to the partial reduction carried out in the main text, in which the discretely parametrized tori were unramified covers Σ_1 over the base.

A.2 Residual Modular Group

The Poincaré normal form is obtained through a change of homology basis of Σ_2. The residual modular group $\mathcal{G}$ is the subgroup of $Sp(4,\mathbb{Z})$ which preserves the form (A.4). It consists of integral matrices of the form

$$\begin{pmatrix} 1 & -\mu\,\alpha & \alpha & \beta \\ 0 & 1-\mu\,\gamma & \gamma & \delta \\ 0 & 0 & 1 & 0 \\ 0 & -\mu^2\,\alpha & \mu\,\alpha & 1+\mu\,\beta \end{pmatrix} \tag{A.9}$$

which obey the $Sp(4,\mathbb{Z})$ condition

$$\gamma - \beta = \mu\,(\alpha\,\delta - \beta\,\gamma)\ . \tag{A.10}$$

The extended fundamental domain $\mathcal{F}_2' = \mathcal{H}_2/\mathcal{G}$ is then constructed as the quotient of the Siegal upper half-plane by the residual modular group.

By using the $Sp(4,\mathbb{Z})$ transformation rule (2.19), one finds that under the action of the residual modular group the period matrix elements transform as

$$\begin{aligned} \Omega_{22} &\longmapsto \frac{\Omega_{22}\,(1-\mu\,\gamma)+\delta}{1+\mu\,\beta-\mu^2\,\alpha\,\Omega_{22}}\ , \\ \Omega_{12} &\longmapsto \frac{\Omega_{12}-\mu\,\alpha\,\Omega_{22}+\beta}{1+\mu\,\beta-\mu^2\,\alpha\,\Omega_{22}}\ , \\ \Omega_{11} &\longmapsto \Omega_{11} + \frac{\alpha\,(\mu\,\Omega_{12}-1)^2}{1+\mu\,\beta-\mu^2\,\alpha\,\Omega_{22}}\ . \end{aligned} \tag{A.11}$$

The Möbius transformations of $\tau = \Omega_{22}$ in the first line of (A.11) form a congruence subgroup of the elliptic modular group $SL(2,\mathbb{Z})$ defined by

$$\Gamma_{(\mu)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \ \middle|\ a,d \equiv 1 \bmod \mu\ ,\ c \equiv 0 \bmod \mu^2\ ,\ b \in \mathbb{Z} \right\} . \tag{A.12}$$

Once the elliptic fundamental domain for $\tau \in \mathcal{H}_1$ is determined, the full genus two fundamental domain $\mathcal{F}_2'$ will follow from the other transformation rules in (A.11).

A.3 Moduli Space

We will now construct a fundamental modular domain in the upper complex half-plane $\mathcal{H}_1$ for the action of the congruence subgroup $\Gamma_{(\mu)} \subset SL(2,\mathbb{Z})$. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ ,\quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{A.13}$$

be the standard generators of $SL(2,\mathbb{Z})$. Consider the fundamental domain Δ for the action of $SL(2,\mathbb{Z})$ given by (2.35), which is a triangle with one vertex at infinity. The three edges separate Δ from the Möbius images $S\bullet\Delta$, $T\bullet\Delta$ and $T^{-1}\bullet\Delta$. A *Schreier transversal* $\mathfrak{C}_{(\mu)}$ for $\Gamma_{(\mu)}$ in $SL(2,\mathbb{Z})$ with respect to $\{S,T\}$ is a set of right coset representatives $SL(2,\mathbb{Z})=\bigcup_g \Gamma_{(\mu)}g$ (i.e. $\Gamma_{(\mu)}g\cap\mathfrak{C}_{(\mu)}$ has precisely one element for each $g\in SL(2,\mathbb{Z})$) expressed as words in the generating set $\{S,T\}$ such that each prefix (or initial segment) of an element of $\mathfrak{C}_{(\mu)}$ is also in $\mathfrak{C}_{(\mu)}$. Then the region

$$\mathfrak{C}_{(\mu)}\bullet\Delta=\bigcup_{C\in\mathfrak{C}_{(\mu)}} C\bullet\Delta \tag{A.14}$$

is a polygonal fundamental domain for the action of $\Gamma_{(\mu)}$ on $\mathcal{H}_1$. For example, if $S\,T\,S\in\mathfrak{C}_{(\mu)}$, then also $S\,T, S, \mathbb{1}_2\in\mathfrak{C}_{(\mu)}$. The triangles $(S\,T\,S)\bullet\Delta$ and $(S\,T)\bullet\Delta$ share a common edge, as do $(S\,T)\bullet\Delta$ and $S\bullet\Delta$, and so on.

Since the subgroup $\Gamma_{(\mu)}\subset SL(2,\mathbb{Z})$ has finite index, there are finite Schreier transversals. The group $\Gamma_{(\mu)}$ is the preimage of the subgroup

$$\phi\big(\Gamma_{(\mu)}\big)=\tilde\Gamma_{(\mu)}:=\left\{\begin{pmatrix}\mu\,a+1 & b\\ 0 & \mu\,d+1\end{pmatrix}\ \middle|\ a+d\equiv 0 \bmod \mu\ ,\ b\in\mathbb{Z}/\mu^2\,\mathbb{Z}\right\} \tag{A.15}$$

of the finite group $SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})$ under the surjective homomorphism

$$\phi\ :\ SL(2,\mathbb{Z})\ \longrightarrow\ SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z}) \tag{A.16}$$

given by reduction modulo μ^2. The index of $\Gamma_{(\mu)}$ in $SL(2,\mathbb{Z})$ may thereby be computed from

$$\begin{aligned}\big[SL(2,\mathbb{Z}):\Gamma_{(\mu)}\big] &= \big[\mathrm{im}(\phi):\tilde\Gamma_{(\mu)}\cap\mathrm{im}(\phi)\big]\\ &= \big[SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z}):\tilde\Gamma_{(\mu)}\big]\ =\ \frac{\big|SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})\big|}{\big|\tilde\Gamma_{(\mu)}\big|}\ .\end{aligned} \tag{A.17}$$

We now need to work out the orders of the two finite groups $SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})$ and $\tilde\Gamma_{(\mu)}$. The order of $\tilde\Gamma_{(\mu)}$ can be easily determined by inspection of its definition (A.15) to be $|\tilde\Gamma_{(\mu)}|=\mu^3$. The order of $SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})$ is calculated as follows.

The index of $\Gamma_{(\mu)}$ turns out to depend crucially on the prime factorization of the integer μ. Suppose that $\mu=p_1^{k(1)}\cdots p_t^{k(t)}$ with p_j, $j=1,\dots,t$ distinct prime numbers and $k(j)>0$. By the Chinese remainder theorem the corresponding finite group factorizes as

$$SL\big(2,\mathbb{Z}/\mu^2\,\mathbb{Z}\big)=SL\big(2,\mathbb{Z}/p_1^{2k(1)}\,\mathbb{Z}\big)\ \times\ \cdots\ \times\ SL\big(2,\mathbb{Z}/p_t^{2k(t)}\,\mathbb{Z}\big) \tag{A.18}$$

and its order is given by

$$\big|SL(2,\mathbb{Z}/\mu^2\,\mathbb{Z})\big|=\prod_{j=1}^t\big|SL(2,\mathbb{Z}/p_j^{2k(j)}\,\mathbb{Z})\big|\ . \tag{A.19}$$

It thus suffices to compute the order of $SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})$ for p prime and $k>0$. Let $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})$. Then $a\,d-b\,c\equiv 1 \bmod p^{2k}$. To ensure that the matrix is non-singular, the pair (a,b) must take values in the set

$$\big(\mathbb{Z}/p^{2k}\,\mathbb{Z}\ \times\ \mathbb{Z}/p^{2k}\,\mathbb{Z}\big)\setminus\big(p\,\mathbb{Z}/p^{2k}\,\mathbb{Z}\ \times\ p\,\mathbb{Z}/p^{2k}\,\mathbb{Z}\big)\ . \tag{A.20}$$

The number of elements in this set is $p^{4k-2}\,(p^2-1)$. The pair (c,d) must be chosen so that p does not divide the determinant. There are $p^{4k-1}\,(p-1)$ such pairs (c,d) for each (a,b). This ensures that the matrix is non-singular. Thus the number of invertible matrices is given by

$$\big|GL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})\big| = p^{4k-2}\,\big(p^2-1\big)\,p^{4k-1}\,\big(p-1\big)\ . \tag{A.21}$$

The determinant is a group homomorphism $\det : GL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z}) \to \mathbb{Z}/p^{2k}\,\mathbb{Z}$. It follows that the index of $SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})$ in $GL(2,\mathbb{Z}/p^{2k}\mathbb{Z})$ is

$$\big[GL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z}) : SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})\big] = \frac{\big|GL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})\big|}{\big|SL(2,\mathbb{Z}/p^{2k}\,\mathbb{Z})\big|} = p^{2k-1}\,(p-1)\ , \tag{A.22}$$

which is just the Euler φ-function of the field $\mathbb{Z}/p^{2k}\,\mathbb{Z}$.

By combining all of these results we find finally that the index of $\Gamma_{(\mu)}$ in $SL(2,\mathbb{Z})$ is given by the Euler product expansion

$$\big[SL(2,\mathbb{Z}) : \Gamma_{(\mu)}\big] = \mu^3 \prod_{\text{primes}\,p|\mu} \left(1-\frac{1}{p^2}\right)\ . \tag{A.23}$$

We can now build a Schreier tranversal inductively, starting from $\{\mathbb{1}_2\}$. Suppose that we have a set $\mathfrak{C}_k$ of k words, satisfying the suffix condition, which contains at most one representative of any right coset. If k is strictly less than the index (A.23), then we can examine the right cosets $\Gamma_{(\mu)}S\,C$, $\Gamma_{(\mu)}T\,C$ and $\Gamma_{(\mu)}T^{-1}\,C$ for each $C\in\mathfrak{C}_k$ until we find one which is different from $\Gamma_{(\mu)}C$ for $C\in\mathfrak{C}_k$. Then add $S\,C$, $T\,C$ or $T^{-1}\,C$ to the list of words to form a new list $\mathfrak{C}_{k+1}$. This process terminates precisely when k is equal to the index (A.23), and then $\mathfrak{C}_k=\mathfrak{C}_{(\mu)}$ is the desired Schreier transversal for $\Gamma_{(\mu)}$. For example, when $\mu=2$ the subgroup $\Gamma_{(2)}$ has index 6 and $\mathfrak{C}_{(2)}=\{\mathbb{1}_2,S,S\,T,S\,T^2,S\,T^3,S\,T^2\,S\}$ is a Schreier transversal for $\Gamma_{(2)}$. The corresponding elliptic fundamental domain (A.14) is depicted in Figure 1.

Using the modular transformations (A.11) along with the positivity constraint (3.24) on the period matrix, we find that the fundamental domain at genus two for the residual modular group preserving the Poincaré normal form is given by

$$\mathcal{F}_2'(\mu) = \big(\mathfrak{C}_{(\mu)}\bullet\Delta\big)\times\mathbb{C}\times\mathcal{H}_1 \tag{A.24}$$

with elements $(\Omega_{22},\Omega_{12},\Omega_{11})$. The integers μ,t,r,a,b,c and d are thus unrestricted except for the dependences of a,b,c and d on r,s and t. Because of this dependence and the complexity of the integration region $\tau\in\mathfrak{C}_{(\mu)}\bullet\Delta$, the free energy cannot be made as explicit as those computed in Sections 3.5, 4.4 and 5.

Appendix B Explicit Form of Ξ_6

In this appendix we provide the explicit expressions for the modular covariant form $\Xi_6[\delta]$ on $\mathcal{H}_2$ defined in (4.14) for the ten even spin structures. Given an even characteristic δ_i, $i=0,1,\dots,9$, we denote $\vartheta_i(\Omega):=\Theta[\delta_i](\mathbf{0}|\Omega)^4$. By the mirror property [19], there are two equivalent expressions for $\Xi_6[\delta_i](\Omega)$ corresponding to the two triples of odd spin structures used to represent $\delta_i=\nu_{i_1}+\nu_{i_2}+\nu_{i_3}=\nu_{i_4}+\nu_{i_5}+\nu_{i_6}$ for each i. One then has

$$\Xi_6[\delta_1] \;=\; -\vartheta_4\,\vartheta_5\,\vartheta_8-\vartheta_2\,\vartheta_6\,\vartheta_9-\vartheta_3\,\vartheta_7\,\vartheta_0 \;=\; -\,\vartheta_4\,\vartheta_7\,\vartheta_6-\vartheta_3\,\vartheta_8\,\vartheta_9-\vartheta_2\,\vartheta_5\,\vartheta_0\ ,$$

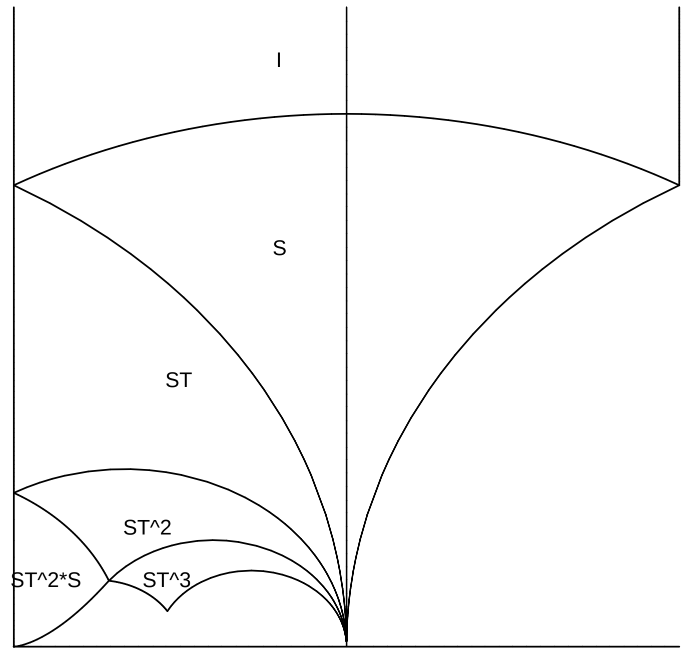

Figure 1: The fundamental domain $\mathfrak{C}_{(2)} \bullet \Delta$ for the action of the congruence subgroup $\Gamma_{(2)} \subset SL(2,\mathbb{Z})$ on $\mathcal{H}_1$.

$$\begin{aligned}
\Xi_6[\delta_2] &= \vartheta_3\,\vartheta_5\,\vartheta_7 + \vartheta_4\,\vartheta_8\,\vartheta_0 - \vartheta_1\,\vartheta_6\,\vartheta_9 \;=\; \vartheta_3\,\vartheta_6\,\vartheta_8 - \vartheta_1\,\vartheta_5\,\vartheta_0 + \vartheta_4\,\vartheta_7\,\vartheta_9 \ , \\
\Xi_6[\delta_3] &= \vartheta_2\,\vartheta_5\,\vartheta_7 - \vartheta_1\,\vartheta_8\,\vartheta_9 + \vartheta_4\,\vartheta_6\,\vartheta_0 \;=\; \vartheta_2\,\vartheta_6\,\vartheta_8 + \vartheta_5\,\vartheta_4\,\vartheta_9 - \vartheta_1\,\vartheta_7\,\vartheta_0 \ , \\
\Xi_6[\delta_4] &= -\vartheta_1\,\vartheta_5\,\vartheta_8 + \vartheta_3\,\vartheta_6\,\vartheta_0 + \vartheta_2\,\vartheta_7\,\vartheta_9 \;=\; -\vartheta_1\,\vartheta_6\,\vartheta_7 + \vartheta_2\,\vartheta_8\,\vartheta_0 + \vartheta_3\,\vartheta_5\,\vartheta_9 \ , \\
\Xi_6[\delta_5] &= \vartheta_2\,\vartheta_3\,\vartheta_7 - \vartheta_1\,\vartheta_4\,\vartheta_6 + \vartheta_6\,\vartheta_9\,\vartheta_0 \;=\; -\vartheta_1\,\vartheta_2\,\vartheta_0 + \vartheta_3\,\vartheta_4\,\vartheta_9 + \vartheta_6\,\vartheta_7\,\vartheta_8 \ , \\
\Xi_6[\delta_6] &= \vartheta_3\,\vartheta_4\,\vartheta_0 - \vartheta_1\,\vartheta_2\,\vartheta_9 + \vartheta_5\,\vartheta_7\,\vartheta_8 \;=\; -\vartheta_1\,\vartheta_4\,\vartheta_7 + \vartheta_5\,\vartheta_9\,\vartheta_0 + \vartheta_2\,\vartheta_3\,\vartheta_8 \ , \\
\Xi_6[\delta_7] &= \vartheta_2\,\vartheta_3\,\vartheta_5 + \vartheta_8\,\vartheta_9\,\vartheta_0 - \vartheta_1\,\vartheta_4\,\vartheta_6 \;=\; \vartheta_2\,\vartheta_4\,\vartheta_9 - \vartheta_1\,\vartheta_3\,\vartheta_0 + \vartheta_5\,\vartheta_6\,\vartheta_8 \ , \\
\Xi_6[\delta_8] &= \vartheta_7\,\vartheta_9\,\vartheta_0 - \vartheta_1\,\vartheta_4\,\vartheta_5 + \vartheta_2\,\vartheta_3\,\vartheta_6 \;=\; -\vartheta_1\,\vartheta_3\,\vartheta_9 + \vartheta_2\,\vartheta_4\,\vartheta_0 + \vartheta_5\,\vartheta_6\,\vartheta_7 \ , \\
\Xi_6[\delta_9] &= \vartheta_7\,\vartheta_8\,\vartheta_0 - \vartheta_1\,\vartheta_2\,\vartheta_6 + \vartheta_3\,\vartheta_4\,\vartheta_5 \;=\; \vartheta_5\,\vartheta_6\,\vartheta_0 - \vartheta_1\,\vartheta_3\,\vartheta_8 + \vartheta_2\,\vartheta_4\,\vartheta_7 \ , \\
\Xi_6[\delta_0] &= \vartheta_7\,\vartheta_8\,\vartheta_9 + \vartheta_3\,\vartheta_4\,\vartheta_6 - \vartheta_1\,\vartheta_2\,\vartheta_5 \;=\; \vartheta_5\,\vartheta_6\,\vartheta_9 + \vartheta_2\,\vartheta_4\,\vartheta_8 - \vartheta_1\,\vartheta_3\,\vartheta_7 \ .
\end{aligned} \tag{B.1}$$

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